PAPER CODE NO.
MATH323

THE UNIVERSITY
of LIVERPOOL

JANUARY 2004 EXAMINATIONS

Bachelor of Science : Year 3
Bachelor of Science : Year 4
Master of Mathematics : Year 3
Master of Mathematics : Year 4
Master of Physics : Year 3
Master of Physics : Year 4
Master of Science : Year 1
No qualification aimed for : Year 1

FURTHER METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

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1. Using the method of variation of arbitrary constants, find the general solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = \frac{e^{2x}}{(x^2+1)} .$$

[**20 marks**]

2. Write down the differential equation satisfied by the function $y(x)$ for which the functional

$$I[y] = \int_a^b F(x, y, y')\, dx \quad \text{with} \quad y(a) = y_0 \ , \ \ y(b) = y_1$$

is stationary, where y_0 and y_1 are constants.
Show that the function $y(x)$ which extremises the functional

$$I[y] = \int_2^4 \left[x^7 \left(y'\right)^2 - 9x^5 y^2 \right] dx$$

satisfies the differential equation

$$x^2 y'' + 7xy' + 9y = 0 .$$

By seeking a solution of the form

$$y = Ax^n + Bx^n \ln x$$

where A, B and n are constants, determine the function which extremises $I[y]$ subject to the boundary conditions, $y(2) = 8$ and $y(4) = 1$.
Evaluate the corresponding extreme value of $I[y]$ and compare it with the value of I for a straight line joining the two endpoints. Establish whether the extremising function maximises or minimises I.

[**20 marks**]

3. Indicate briefly how you would find the function $y(x)$ satisfying $y(a) = y_0$ and $y(b) = y_1$, such that the functional

$$I[y] \;=\; \int_a^b F(x, y, y')\, dx$$

is stationary subject to the condition that a second functional

$$J[y] \;=\; \int_a^b G(x, y, y')\, dx$$

is equal to a constant.
A chain of length L hangs below the x-axis between the points $(0, 0)$ and $(4, 0)$ in the (x, y) plane, where the positive y direction is in the same sense as gravity, in such a way that its potential P is minimised where

$$P \;=\; \int_0^4 y \left[1 \;+\; (y')^2\right]^{1/2} dx$$

and

$$L \;=\; \int_0^4 \left[1 \;+\; (y')^2\right]^{1/2} dx\ .$$

Show that the extremal curve satisfies the differential equation

$$\frac{dy}{dx} \;=\; \left[\left(\frac{y+\alpha}{\beta}\right)^2 \;-\; 1\right]^{1/2}$$

where α and β are constants.
If $L = 6$ find the extremal curve.

[You may assume that $\cosh x$ is an even function and that the numerical solution to the equation

$$x \sinh\left(\frac{2}{x}\right) \;=\; 3$$

is $x \;=\; -\,1.23295$.]

[20 marks]

4. The functions $u(x,y)$ and $v(x,y)$ satisfy the simultaneous partial differential equations

$$\begin{aligned} v_x + 4xu_y &= 0 \\ u_x + xv_y &= 0 \,. \end{aligned}$$

Show that this system of differential equations is hyperbolic with characteristics

$$\begin{aligned} y - x^2 &= \eta = \text{constant} \\ y + x^2 &= \nu = \text{constant} \,. \end{aligned}$$

Hence show that the Riemann invariants of the system are

$$2u + v = \text{constant} \quad \text{and} \quad v - 2u = \text{constant} \,.$$

Find solutions $u(x,y)$ and $v(x,y)$ such that

$$u(x,0) = \frac{x^2}{6}(x^2-1) \quad \text{and} \quad v(x,0) = -\frac{2}{3}x^2 \,.$$

[20 marks]

5. Show that the partial differential equation satisfied by $u(x,y)$,

$$x^2u_{xx} - 4y^2u_{yy} = F(x,y,u_x,u_y)$$

with $x \neq 0$ and $y \neq 0$, is hyperbolic with characteristics

$$\eta(x,y) = \frac{y}{x^2} = \text{constant} \quad \text{and} \quad \nu(x,y) = x^2y = \text{constant} \,.$$

Show that the canonical form of this partial differential equation is

$$-16\eta\nu u_{\eta\nu} + 6\eta u_\eta + 2\nu u_\nu = F \,.$$

Find the general solution when

$$F = \frac{x}{2}u_x + yu_y \,.$$

[20 marks]

6. Show that the Joukowski transformation, where $z \neq 0$,

$$w \;=\; u \;+\; iv \;=\; z \;+\; \frac{1}{z}$$

is conformal for all $z \neq \pm 1$, where u and v are real and w and z are complex variables.
Find u and v in terms of polar coordinates r and θ where $z = re^{i\theta}$.
Show that the circles $|z| = 2$ and $|z| = \frac{1}{2}$ are mapped to the same ellipse,

$$E_1 : \quad 9u^2 \;+\; 25v^2 \;=\; \frac{225}{4} \;,$$

in the w-plane. Hence show that the exterior of $|z| = 2$ is mapped to the exterior of E_1.
The potential Φ outside the circular region $|z| = 2$ is given by

$$\Phi \;=\; \mathrm{Re}\,(A \ln z) \;=\; A \ln r$$

where A is a real constant. Show that Φ satisfies Laplace's equation in polar coordinates:

$$\frac{\partial^2 \Phi}{\partial r^2} \;+\; \frac{1}{r}\frac{\partial \Phi}{\partial r} \;+\; \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} \;=\; 0 \;.$$

By clearly stating your reasoning, show that the potential outside the ellipse E_1 is given by

$$\Phi \;=\; A\,\mathrm{Re}\,\ln\left[\frac{w + \sqrt{(w^2 - 4)}}{2}\right] \;.$$

Determine the real and imaginary parts of the potential at the point $w = 3i$.

[20 marks]

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7. The Fourier transform of a function $f(x)$ suitably defined on $-\infty < x < \infty$ and its inverse are respectively

$$F\left(f(x);\omega\right) = \bar{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

$$F^{-1}(\bar{f}(\omega);x) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\omega)e^{i\omega x}\,dx\ .$$

By using integration by parts show that when $f(x) = e^{-a^2x^2}$ its Fourier transform, $\bar{f}(\omega)$, satisfies the first order differential equation

$$\bar{f}'(\omega) = -\frac{\omega}{2a^2}\bar{f}(\omega)$$

where a is a real non-zero constant.
By considering the square of $\bar{f}(0)$ show that for $f(x) = e^{-a^2x^2}$

$$\bar{f}(0) = \frac{\sqrt{\pi}}{a}\ .$$

Hence show that

$$F\left(e^{-a^2x^2};\omega\right) = \frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}\ .$$

The function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

for $t \geq 0$ subject to the initial conditions

$$u(x,0) = g(x)\quad ,\quad u, u_x \to 0 \quad \text{as } x \to \pm\infty$$

where c is a constant.
By using a Fourier transform show that $u(x,t)$ is given by

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} g\left(x - 2cz\sqrt{t}\right)e^{-z^2}\,dz\ .$$

[You may assume that the convolution theorem for the product of two Fourier transforms, $\bar{g}(\omega)$ and $\bar{h}(\omega)$, is

$$\int_{-\infty}^{\infty} \bar{g}(\omega)\bar{h}(\omega)e^{i\omega x}\,d\omega = 2\pi\int_{-\infty}^{\infty} g(z)h(x-z)\,dz\ .\Big]$$

[20 marks]