PAPER CODE NO.
MATH323

THE UNIVERSITY
of LIVERPOOL

JANUARY 2003 EXAMINATIONS

Bachelor of Science	:	Year 3
Bachelor of Science	:	Year 4
Master of Mathematics	:	Year 3
Master of Mathematics	:	Year 4

FURTHER METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

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1. By seeking a solution of the form $y = Ax^n$ where A and n are constants, find the general solution of

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0 .$$

Show that the same method does not generate the general solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 0 .$$

Verify that the general solution in this case is in fact

$$y(x) = c_1 x^2 + c_2 x^2 \ln x$$

where c_1 and c_2 are arbitrary constants.
Using the method of variation of arbitrary constants, find the general solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2 .$$

[20 marks]

2. Write down the differential equation satisfied by the function $y(x)$ for which the functional

$$I[y] = \int_a^b F(x, y')\, dx \quad \text{with} \quad y(a) = y_0 \, , \quad y(b) = y_1$$

is stationary, where y_0 and y_1 are constants.
Given

$$I[y] = \int_{\frac{1}{2}}^{1} y' \left[1 + x^3 y'\right] dx \quad \text{with} \quad y(\tfrac{1}{2}) = 0 \, , \quad y(1) = 3$$

show that the extremising function satisfies

$$\frac{dy}{dx} = \frac{A}{x^3} .$$

where A is a constant.
Integrate this equation and use the boundary conditions to determine the extremal $y(x)$ and calculate the corresponding extreme value of $I[y]$.
Compare this result with the value of I obtained for a straight line joining the endpoints. What is the nature of the extremum?

[20 marks]

3. A dynamical system with one degree of freedom $q(t)$ is described by the Lagrangian $L(q, \dot{q}, t)$ where $\dot{q} = dq/dt$. Write down Lagrange's equation.
The Hamiltonian H is defined by

$$H(q, p, t) \; = \; p\dot{q} \; - \; L$$

where $p = \partial L/\partial \dot{q}$. Given Lagrange's equation show that Hamilton's equations,

$$\frac{\partial H}{\partial q} \; = \; - \; \dot{p} \quad , \quad \frac{\partial H}{\partial p} \; = \; \dot{q} \; ,$$

follow.
A particle of mass m in motion on the x-axis in a potential $V(x)$ is described by the Hamiltonian

$$H(x, p, t) \; = \; \frac{p^2}{2m} e^{-at} \; + \; V(x, t)$$

where a is a positive constant. Write down Hamilton's equations for the system.
In the case when $V(x, t) = kx\cos(\omega t)$ find $x(t)$ when the initial conditions are $x(0) = 0$ and $p(0) = p_0$. Describe what happens to the particle as $t \to \infty$.

[20 marks]

4. The functions $u(x, y)$ and $v(x, y)$ satisfy the simultaneous partial differential equations

$$\begin{aligned} u_x \; - \; 2u_y \; - \; 3v_y \; &= \; 0 \\ 3u_x \; + \; 2v_x \; + \; 2v_y \; &= \; 0 \; . \end{aligned}$$

Show that this system of differential equations is hyperbolic with characteristics

$$\begin{aligned} x \; + \; 2y \; &= \; \alpha \; = \; \text{constant} \\ 4x \; - \; y \; &= \; \beta \; = \; \text{constant} \; . \end{aligned}$$

Hence show that the Riemann invariants of the system are

$$u \; + \; 2v \; = \; \text{constant} \qquad \text{and} \qquad 2u \; + \; v \; = \; \text{constant} \; .$$

Find solutions $u(x, y)$ and $v(x, y)$ such that $u(0, y) = y^3$ and $v(0, y) = - \; y$.

[20 marks]

5. Show that the partial differential equation satisfied by $u(x, y)$,

$$x^2 u_{xx} \;-\; 4x u_{xy} \;+\; 4u_{yy} \;+\; 2u_y \;=\; 2x^3$$

where $x \neq 0$, is parabolic with characteristics

$$\eta(x, y) \;=\; y \;+\; 2\ln x \;=\; \text{constant} \qquad \text{and} \qquad \nu(x, y) \;=\; \text{constant}$$

where $\nu(x, y)$ satisfies $x\nu_x \neq 2\nu_y$, justifying the origin of this condition.
By choosing $\nu = x$ reduce the partial differential equation to the canonical form

$$\frac{\partial^2 u}{\partial \nu^2} \;=\; 2\nu$$

and find its general solution in terms of x and y.
Determine the particular solution if

$$u(x, 0) \;=\; \frac{x^3}{3} \;+\; 4(x-1)(\ln x)^2 \quad \text{and} \quad u(1, y) \;=\; \frac{1}{3}\,.$$

[**20 marks**]

6. Show that the most general solution of Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2} \;+\; \frac{\partial^2 \Phi}{\partial y^2} \;=\; 0$$

which is independent of y is given by $\Phi \;=\; Ax \;+\; B$ where A and B are constants. Determine A and B given that $\Phi = 2$ when $x = 1$ and $\Phi = -\,2$ when $x = 2$.
Verify that the transformation $w \;=\; e^{\pi z}$ is conformal for all z where $w \;=\; u \;+\; iv$ and $z \;=\; x \;+\; iy$. Show that $w \;=\; e^{\pi z}$ maps the interior of the rectangle with vertices $(1, 1)$, $(2, 1)$, $(2, -1)$ and $(1, -1)$ into the region bounded by concentric circles centred on the origin of radii e^{π} and $e^{2\pi}$. Sketch both regions indicating clearly where the vertices are mapped to in the w-plane.
Hence determine the solution to Laplace's equation, $\Phi(u, v)$, in the region between the circles

$$u^2 \;+\; v^2 \;=\; e^{2\pi} \qquad \text{and} \qquad u^2 \;+\; v^2 \;=\; e^{4\pi}$$

in the (u, v)-plane given that $\Phi = 2$ on the inner circle and $\Phi = -\,2$ on the outer circle.

[**20 marks**]

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7. The Fourier sine transform of a function $f(x)$ defined on $0 < x < \infty$ is

$$F_s\left(f(x);\omega\right) \;=\; \bar{f}_s(\omega) \;=\; \int_0^\infty f(x)\sin(\omega x)\,dx$$

where the inverse transform is

$$f(x) \;=\; \frac{2}{\pi}\int_0^\infty \bar{f}_s(\omega)\sin(\omega x)\,d\omega \ .$$

Show that for a function $f(x)$ such that both $f(x) \to 0$ and $f'(x) \to 0$ as $x \to \infty$

$$F_s\left(f''(x);\omega\right) \;=\; -\ \omega^2 F_s(f(x);\omega) \ +\ \omega f(0) \ .$$

The function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} \;=\; k\frac{\partial^2 u}{\partial x^2}$$

where $0 < x < \infty$ and $0 < t < \infty$ and is subject to the boundary conditions $u(0,t) = A$ and $u(x,0) = 0$ where A is constant. Show, by taking the Fourier sine transform, that

$$u(x,t) \;=\; \frac{2A}{\pi}\int_0^\infty \frac{d\omega}{\omega}\left[1 \ -\ e^{-k\omega^2 t}\right]\sin(\omega x) \ .$$

Determine the value of $u(x,t)$ as $t \to \infty$.

[You may use the result

$$\int_0^\infty dx\,\frac{\sin x}{x} \;=\; \frac{\pi}{2}$$

without derivation.]

[20 marks]