PAPER CODE NO.
MATH323

THE UNIVERSITY
of LIVERPOOL

JANUARY 2002 EXAMINATIONS

Bachelor of Science : Year 3
Bachelor of Science : Year 4
Master of Mathematics : Year 3

FURTHER METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

1. Using the method of variation of arbitrary constants find the general solution of the following differential equations

$$\text{(i)} \quad y'' - 4y' + 3y = 2xe^{2x}$$

$$\text{(ii)} \quad y''' - y' = 4\cos x \ .$$

[You may use without derivation the result

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{(a^2+b^2)}\left[a\cos(bx) + b\sin(bx)\right]$$

for real constants a and b.]

[20 marks]

2. Write down the differential equation satisfied by the function $y(x)$ for which the functional

$$I[y] = \int_a^b F(x, y, y')\,dx \quad \text{with} \quad y(a) = y_0 \ , \quad y(b) = y_1$$

is stationary, where y_0 and y_1 are constants.
Show that the function $y(x)$ which extremises the functional

$$I[y] = \int_1^2 \left[x^2 y'^2 + 12y^2\right] dx \quad \text{with} \quad y(1) = 1 \ , \quad y(2) = 8$$

must satisfy the equation

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = 0 \ .$$

By seeking a solution of the form $y = Ax^n$ where A and n are constants, determine the extremal curve $y(x)$ and evaluate the corresponding extreme value of I. Compare this result to the value of I obtained for a straight line joining the endpoints. What is the nature of the extremum?

[20 marks]

3. Indicate briefy how you would find the function $y(x)$ satisfying $y(a) = y_0$ and $y(b) = y_1$, such that the functional

$$I[y] \;=\; \int_a^b F(x, y, y')\, dx$$

is stationary subject to the condition that a second functional

$$J[y] \;=\; \int_a^b G(x, y, y')\, dx$$

is equal to a constant.
For the case

$$I[y] \;=\; \int_0^{\pi/2} \left[y'^2 \;-\; y^2 \right] dx$$

and

$$J[y] \;=\; \int_0^{\pi/2} y\, dx \;=\; 1$$

where $y(0) = 0$ and $y(\pi/2) = 0$, find the extremal curve.
[You need not evaluate the corresponding extreme value of I.]

[20 marks]

4. The functions $u(x, y)$ and $v(x, y)$ satisfy the simultaneous partial differential equations

$$\begin{aligned} 3u_x \;+\; u_y \;+\; 2v_y \;&=\; 0 \\ 11u_x \;+\; 2v_x \;+\; 3v_y \;&=\; 0\,. \end{aligned}$$

Show that this system of differential equations is hyperbolic with characteristics

$$\begin{aligned} x \;+\; 3y \;&=\; \alpha \;=\; \text{constant} \\ 3x \;+\; 2y \;&=\; \beta \;=\; \text{constant}\,. \end{aligned}$$

Hence show that the Riemann invariants of the system are

$$u \;+\; v \;=\; \text{constant} \qquad \text{and} \qquad 11u \;+\; 4v \;=\; \text{constant}\,.$$

Find solutions $u(x, y)$ and $v(x, y)$ such that $u(x, 0) = 2x^2$ and $v(x, 0) = -\,2x^2$.

[20 marks]

5. Show that the partial differential equation satisfied by $u(x, y)$,

$$4u_{xx} + 8u_{xy} + 3u_{yy} = 8u_x + 12u_y ,$$

is hyperbolic with characteristics which may be chosen as

$$\eta = 3x - 2y \quad \text{and} \quad \nu = x - 2y .$$

Hence show that the original partial differential equation may be written in canonical form as

$$\frac{\partial^2 u}{\partial\eta\partial\nu} = \frac{\partial u}{\partial\nu}$$

and find its general solution in terms of x and y. [**20 marks**]

6. Show that the most general solution of Laplace's equation

$$\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2} = 0$$

which is independent of v is given by $\Phi = A + Bu$ where A and B are constants. Determine A and B given that $\Phi = 1/12$ when $u = 2$ and $\Phi = 1/6$ when $u = 3$.

Show that that the transformation $w = 12/z$ where $w = u + iv$ and $z = x + iy$ maps the circles

$$C_1 : \quad |z - 3| = 3$$
$$C_2 : \quad |z - 2| = 2$$

into the lines

$$L_1 : \quad u = 2$$
$$L_2 : \quad u = 3$$

respectively. Sketch the two circles and by considering the point $z = 5$ show that the region between the two circles maps into the region between the two lines.

Using the previous results find $\Phi(x, y)$ given that Φ satisfies Laplace's equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0$$

in the region between C_1 and C_2 with $\Phi = 1/12$ on C_1 and $\Phi = 1/6$ on C_2. [**20 marks**]

7. The Fourier transform of a function $f(x)$ defined on $-\infty < x < \infty$ is

$$F\left(f(x);\omega\right) \;=\; \bar{f}(\omega) \;=\; \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx\ .$$

Show that for a function $f(x)$ such that $f(x) \to 0$ as $x \to \pm\infty$

$$F\left(\frac{df(x)}{dx};\omega\right) \;=\; i\omega\bar{f}(\omega)$$

and write down the Fourier transform of the nth derivative of $f(x)$ where n is a positive integer.
If the Fourier transform of $f(x) = e^{-a|x|}$ is $\bar{f}(\omega) = 2a/(a^2+\omega^2)$, show that the Fourier transform of

$$g(x) \;=\; \frac{1}{(a^2+x^2)}$$

is

$$\bar{g}(\omega) \;=\; \frac{\pi}{a}e^{-a|\omega|}\ .$$

The function $u(x,y)$ satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \;=\; 0$$

for $y \geq 0$ such that $u \to 0$ as $y \to \infty$ and $u(x,0) = h(x)$. By using a Fourier transform show that

$$u(x,y) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{h}(\omega)e^{-|\omega|y}e^{i\omega x}\,d\omega\ .$$

Hence, using the convolution theorem show that

$$u(x,y) \;=\; \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{h(\xi)}{[(x-\xi)^2+y^2]}\,d\xi\ .$$

[You may use the result

$$\int_{-\infty}^{\infty} e^{i\omega(x-y)}\,d\omega \;=\; 2\pi\delta(x-y)$$

where $\delta(x-y)$ is the Dirac δ-function.]

[20 marks]