M323 JAN 2001

Instructions to candidates

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

1. Describe briefly the Method of Variation of Arbitrary Constants for the solution of second-order ordinary linear differential equations.

[6 marks]

Verify that the general solution of the differential equation

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

is given by

$$y = C_1 x + C_2 e^x$$

where C_1 and C_2 are arbitrary constants.

[3 marks]

Using the Method of Variation of Arbitrary Constants, show that the solution to the equation

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = f(x)$$

may be written in the form

$$y = L_1(x)x + L_2(x)e^x$$

where

$$\frac{dL_1}{dx} = -\frac{f(x)}{(1-x)^2}$$
$$\frac{dL_2}{dx} = \frac{xf(x)e^{-x}}{(1-x)^2}.$$

[4 marks]

Hence find the general solution for the case

$$f(x) = -(1-x)^2 e^x \cos x.$$

[7 marks]

2. Write down the differential equation satisfied by the function $y(x)$ for which the functional

$$I[y] = \int_a^b F(x, y, y')\,dx \qquad y(a) = y_0, \quad y(b) = y_1$$

is stationary. Show that if F is not an explicit function of y then the extremal curve satisfies the equation

$$\frac{\partial F}{\partial y'} = C$$

where C is a constant.

[3 marks]

For the case

$$I[y] = \int_0^1 (y' + (1+x)y'^2)\,dx \qquad y(0) = 0, \quad y(1) = \ln 2,$$

show that the extremal curve satisfies the equation

$$\frac{dy}{dx} = \frac{A}{1+x}$$

where A is a constant. Hence find the extremal curve. Calculate $I[y]$ both for the extremal curve and for the straight line joining the two end points. What do you think is the nature of the extremum?

[17 marks]

3. Indicate briefly how you would find the function $y(x)$, satisfying $y(a) = y_0, \quad y(b) = y_1$, such that the functional

$$I[y] = \int_a^b F(x, y, y')\, dx$$

is stationary, subject to the condition that a second functional

$$J[y] = \int_a^b G(x, y, y')\, dx$$

is equal to a constant.

[3 marks]

For the case

$$I[y] = \int_0^1 xy\, dx$$

and

$$J[y] = \int_0^1 (y')^2\, dx = \frac{4}{5}$$

where $y(0) = y(1) = 0$, find the extremal curves. Hence find the maximum possible value of I.

[17 marks]

4. The functions $u(x, y)$ and $v(x, y)$ satisfy the simultaneous partial differential equations

$$\frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0$$

$$3\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + 8\frac{\partial v}{\partial y} = 0.$$

Show that this system of differential equations is hyperbolic, with characteristics which may be chosen as follows

$$\alpha = y - 6x$$

$$\beta = y + 4x.$$

[6 marks]

Hence show by changing variables $(x, y) \to (\alpha, \beta)$ that u and v satisfy the equations:

$$9\frac{\partial u}{\partial \alpha} - \frac{\partial v}{\partial \alpha} = 0$$

$$\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \beta} = 0.$$

[7 marks]

Find the general solution for u and v in terms of x and y. Find also the solution that satisfies the boundary conditions $u(x, 0) = \sin x$ and $v(x, 0) = \cos x$.

[7 marks]

5. The function $u(x,y)$ satisfies the partial differential equation

$$x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = F(x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}), \quad x \neq 0, y \neq 0. \qquad (1)$$

Show that this equation is parabolic and that it can be reduced to canonical form by the change of variables

$$\nu = xy \quad \text{and} \quad \eta = \eta(x,y),$$

where η is such that

$$y\frac{\partial \eta}{\partial y} \neq x\frac{\partial \eta}{\partial x}.$$

[8 marks]

Choosing $\eta = x$, reduce Eq. (1) to canonical form.

[8 marks]

For the case

$$F = x^2 - 2y\frac{\partial u}{\partial y},$$

show that u satisfies the equation

$$\frac{\partial^2 u}{\partial \eta^2} = 1.$$

Hence show that the general solution of Eq. (1) is

$$u(x,y) = g(xy) + xf(xy) + \frac{1}{2}x^2,$$

where f and g are arbitrary functions.

[4 marks]

6. (i) Verify that $\Phi = 1 - \frac{1}{\pi}\phi$, where $w = u + iv = \rho e^{i\phi}$, is a function harmonic in the upper-half w-plane, taking the values $\Phi = 0$ for $u < 0, v = 0$ and $\Phi = 1$ for $u > 0, v = 0$. Show that in terms of u and v this solution takes the form

$$\begin{cases} \Phi = 1 - \frac{1}{\pi}\tan^{-1}\frac{v}{u} & u > 0 \\ \Phi = -\frac{1}{\pi}\tan^{-1}\frac{v}{u} & u < 0. \end{cases}$$

[7 marks]

(ii) Show that the conformal transformation

$$w = i\left(\frac{1-z}{1+z}\right)$$

maps the interior of the unit circle $|z| = 1$ into the upper half w-plane, $\operatorname{Im} w > 0$. Indicate in a diagram how the two semicircles $|z| = 1, \operatorname{Im} z > 0$ and $|z| = 1, \operatorname{Im} z < 0$ are mapped into the w-plane.

[8 marks]

(iii) Using (i) and (ii), construct a function harmonic inside the unit circle $|z| = 1$, taking prescribed values $F(\theta)$ on its circumference as follows:

$$\begin{aligned} F(\theta) &= 1 \quad \text{for} \quad 0 < \theta < \pi \\ &= 0 \quad \text{for} \quad \pi < \theta < 2\pi. \end{aligned}$$

[5 marks]

7. The Fourier Transform of a function $f(x)$ defined on the interval $-\infty < x < \infty$ is

$$F\{f(x);\omega\} = \overline{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx.$$

Show that if $f \to 0$ when $x \to \pm\infty$, then

$$F\{\frac{df}{dx};\omega\} = i\omega\overline{f}(\omega).$$

[4 marks]

The function $g(x)$ is defined as follows:

$$\begin{cases} g(x) = 1 & -b \le x \le b \\ g(x) = 0 & |x| > b \end{cases}$$

where b is a positive constant. Show that

$$\overline{g}(\omega) = \frac{2\sin(b\omega)}{\omega}.$$

[3 marks]

The function $U(x,t)$ satisfies the partial differential equation

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}$$

for $-\infty < x < \infty$ and $t > 0$, where $k > 0$. U also satisfies the following conditions:

$$U(x,0) = g(x) \quad \text{and} \quad U, \frac{\partial U}{\partial x} \to 0 \quad \text{as} \quad x \to \pm\infty.$$

Using a Fourier transform, show that

$$U(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(b\omega)e^{-k\omega^2 t}}{\omega} e^{i\omega x}\,d\omega.$$

[9 marks]

Using the convolution formula:

$$F^{-1}\{\overline{f}(\omega)\overline{g}(\omega)\} = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi,$$

or otherwise, show that an alternative expression for $U(x,t)$ is

$$U(x,t) = \frac{a}{\sqrt{\pi}}\int_{-b}^{b} e^{-a^2(x-\xi)^2}\,d\xi$$

where $a^2 = 1/(4kt)$. [4 marks]

[You may use without derivation the result

$$F\{e^{-a^2x^2};\omega\} = \frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}},$$

where a is a positive constant.]