M323 JAN 2000

Instructions to candidates

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

1. Write down the differential equation satisfied by the function $y(x)$ for which the functional

$$I[y] = \int_a^b F(x, y, y')\, dx \qquad y(a) = y_a, \quad y(b) = y_b$$

is stationary.

Show that if F is not an explicit function of x then the extremal curve satisfies the equation

$$F - y' \frac{\partial F}{\partial y'} = C$$

where C is a constant.

For the case

$$F = \frac{(1 + y'^2)^{\frac{1}{2}}}{y + 1}$$

show that the extremal curve satisfies the equation

$$(1 + y'^2)^{\frac{1}{2}} (y + 1) = c_1$$

where c_1 is a constant. By substituting $y' = \tan\theta$ (or otherwise) show that the equation of the extremal may be written in parametric form

$$y = c_1 \cos\theta - 1$$
$$x = -c_1 \sin\theta + c_2$$

where c_2 is another constant, and show by eliminating θ that the extremal is an arc of a circle.

Given $a = 0, b = 1, y_a = 0, y_b = 1$, find c_1 and c_2 and hence the centre and radius of the circle.

2. A dynamical system with one degree of freedom $q(t)$ is described by the Lagrangian $L(q, \dot{q}, t)$, where $\dot{q} \equiv \frac{dq}{dt}$. Write down Lagrange's equation.

The Hamiltonian H is defined as follows:

$$H(q, p, t) = p\dot{q} - L$$

where $p = \frac{\partial L}{\partial \dot{q}}$. Show that, given Lagrange's equation, it follows that Hamilton's equations:

$$\frac{\partial H}{\partial q} = -\dot{p} \quad \text{and} \quad \frac{\partial H}{\partial p} = \dot{q}$$

are also true.

A particle of mass m in motion on the x-axis is described by the Hamiltonian:

$$H(x, p, t) = \frac{p^2}{2m} + kx \sin \omega t$$

where k and ω are constants. Write down Hamilton's equations for the system. Given initial conditions $x(0) = p(0) = 0$, show that the position $x(t)$ of the particle is given by

$$x = \frac{kt}{m\omega} \left[\frac{\sin \omega t}{\omega t} - 1 \right].$$

Prove that the particle never returns to the origin.

3. The functions $u(x, y)$ and $v(x, y)$ satisfy the simultaneous partial differential equations

$$\frac{\partial u}{\partial x} + 7\frac{\partial v}{\partial x} + 5\frac{\partial v}{\partial y} = 5x$$
$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = x.$$

Show that this system of differential equations is hyperbolic, with characteristics which may be chosen as follows:

$$y - 5x = \alpha = \text{constant}$$
$$y - x = \beta = \text{constant}.$$

Hence show that the Riemann invariants of the system are

$$u + 6v - 2x^2 = \text{constant (when } \beta \text{ is constant), and}$$
$$u + 2v = \text{constant, (when } \alpha \text{ is constant).}$$

Hence write down the general solution for $u(x, y)$ and $v(x, y)$.

4. Show that the partial differential equation

$$2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 3\frac{\partial z}{\partial x} + 6\frac{\partial z}{\partial y} \qquad (1)$$

is hyperbolic, with characteristics which may be chosen to be

$$\nu = y - 2x \quad \text{and} \quad \eta = 2y - x.$$

Hence show that Equation (1) may be written in canonical form as:

$$\frac{\partial^2 z}{\partial \eta \partial \nu} = -\frac{\partial z}{\partial \eta},$$

and find its general solution in terms of x and y.

5.

(i) Show that the most general solution of Laplace's equation

$$\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = 0$$

that is independent of u is given by

$$\Phi = A + Bv$$

where A and B are constants. Determine A, B given that $\Phi = 3$ when $v = 1$ and $\Phi = 5$ when $v = 2$.

(ii) Show that the transformation

$$w = \frac{4}{z},$$

where $w = u + iv$ and $z = x + iy$, maps the circles

$$C_1 : \quad |z + i| = 1 \quad \text{and}$$
$$C_2 : \quad |z + 2i| = 2$$

into the lines $L_1 : v = 2$, and $L_2 : v = 1$ respectively. Sketch the two circles, and by considering the point $z = -3i$, show that the region between the two circles maps into the region between the two lines.

(iii) Using the results of (i) and (ii), find $\Phi(x, y)$ given that Φ satisfies Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

in the region between C_1 and C_2, with $\Phi = 5$ on C_1 and $\Phi = 3$ on C_2.

6. The Fourier Transform of a function $f(x)$ defined on the interval $-\infty < x < \infty$ is

$$F\{f(x);\omega\} = \overline{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx.$$

Show that the Fourier Transform of the function $g(x) = e^{-a|x|}$, where a is a positive constant, is given by

$$\overline{g}(\omega) = \frac{2a}{a^2+\omega^2}.$$

Show also that for a function $f(x)$ such that $f \to 0$ when $x \to \pm\infty$,

$$F\{\frac{df}{dx};\omega\} = i\omega\overline{f}(\omega).$$

The function $\phi(x,y)$ satisfies Laplace's equation:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

for $0 \le y \le h$, with boundary conditions as follows:

$$\phi(x,0) = 0, \quad \phi(x,h) = e^{-a|x|}, \quad \phi, \phi_x, \to 0 \text{ as } x \to \pm\infty.$$

By using a Fourier Transform, show that $\phi(x,y)$ is given by:

$$\phi(x,y) = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega x}\sinh(\omega y)}{(a^2+\omega^2)\sinh(\omega h)}\,d\omega.$$

7. By seeking a solution of the form $y = Ax^n$ where A and n are constants, find the general solution of the equation

$$x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} - 16y = 0.$$

Show that the same method does not generate the general solution of the equation

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0.$$

Verify that the general solution in this case is in fact

$$y = c_1x + c_2x\ln x$$

where c_1 and c_2 are arbitrary constants.

Using the the Method of Variation of Arbitrary Constants (and deriving any formulae you use associated with the Method), find the general solution of the differential equation

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \ln x.$$