THE UNIVERSITY
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Math 322

May 2006 exam

Solutions

Ref. M322exam06_final_sols_issue.tex

Note:- final version for issue. 28/6/06

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1. (a) $f(x)$ is a continuous, differentiable and invertible function with domain $[0, 1]$ and range $[0, 1]$. $f(x)$ has only three fixed points: unstable fixed points at $x = 0$ and 1, and a stable fixed point at $x = 0.5$.

(i) Sketch the graph of $f(x)$. [2 marks]

(ii) Find the basin of attraction for the stable fixed point and indicate it on the graph. [1 marks]

(b) A tent map $f(x)$ for x in $[0, 1]$ is defined as

$$f(x) = 3\mu x \quad \text{for} \quad x \leq \frac{1}{2}$$

$$f(x) = 3\mu(1 - x) \quad \text{for} \quad x > \frac{1}{2}$$

where $0 \leq \mu \leq 1$.

Consider the dynamical system given by iterations of this map

$$x_{n+1} = f(x_n).$$

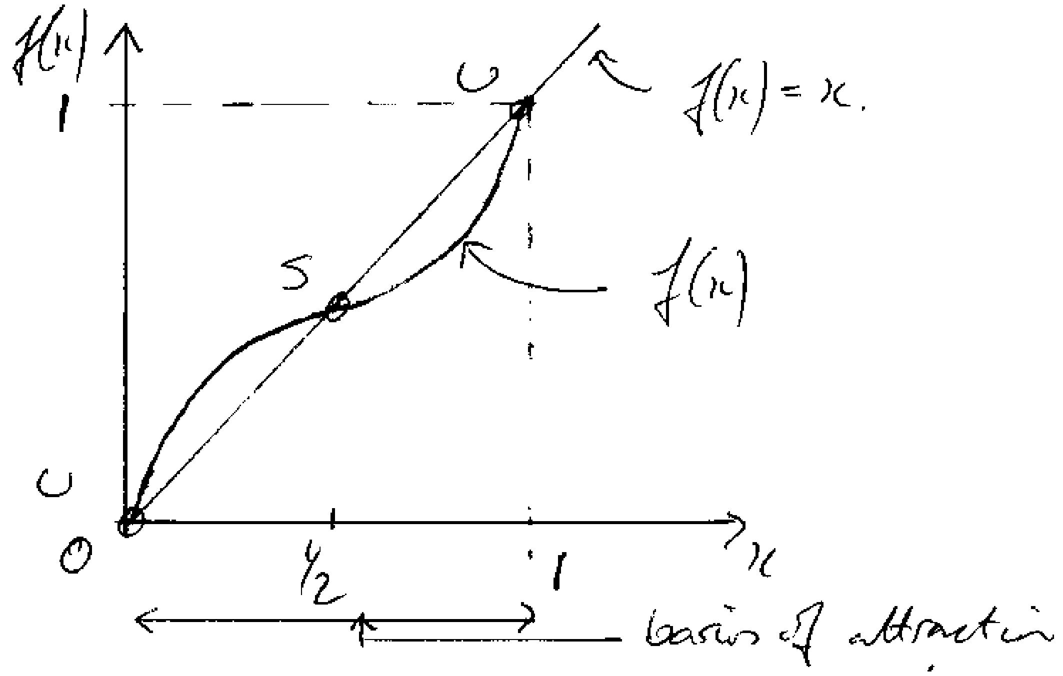
(i) Sketch the graph of the function $f(x)$ for the two cases $\mu < 1/3$ and $\mu > 1/3$. [3 marks]

(ii) Show that, provided that $\mu \neq 1/3$, the fixed points are $x^* = 0$ for any μ and $x^* = 3\mu/(1 + 3\mu)$ if $\mu > 1/3$. [7 marks]

(iii) Show that $x^* = 0$ is a stable fixed point for $\mu < 1/3$ and that $x^* = 3\mu/(1 + 3\mu)$ is an unstable fixed point. [7 marks]

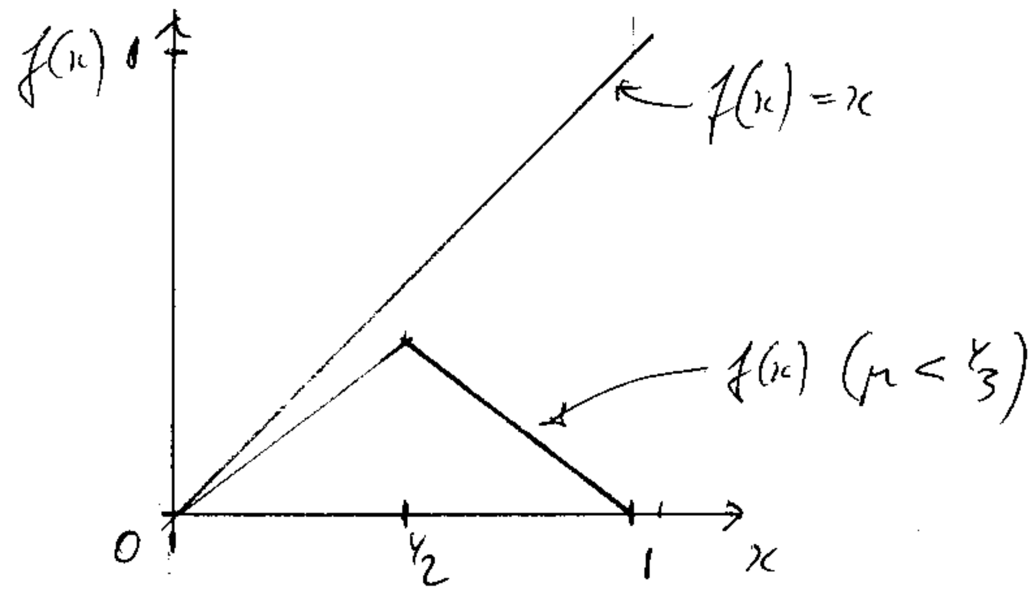
Solution

(a) (i) Since the function is continuous, differentiable and invertible, with domain and range $[0, 1]$, it must be as below. [2 marks]

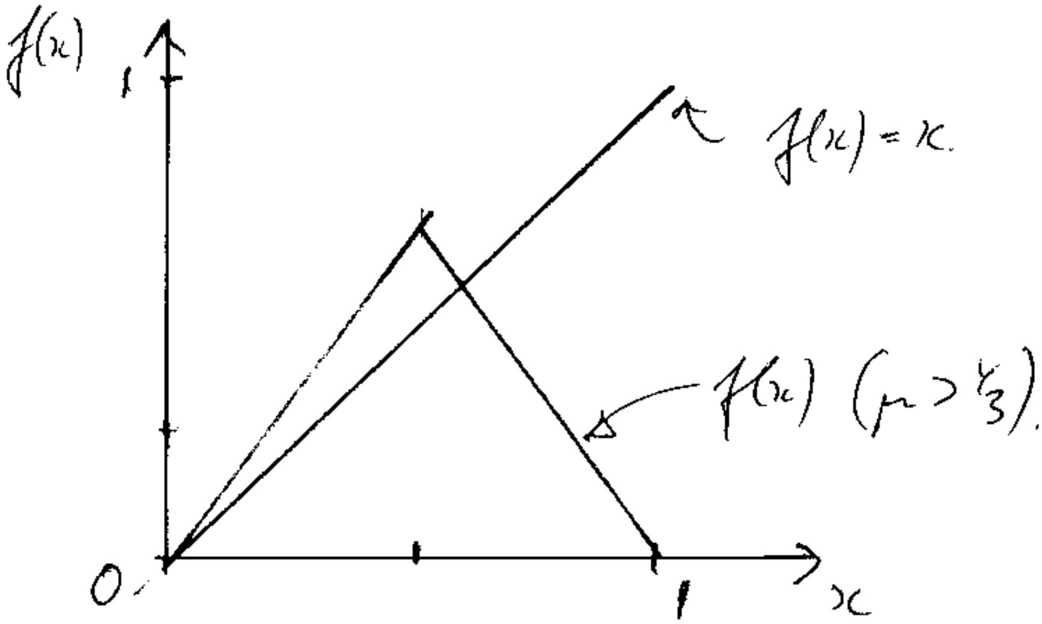


(ii) Since the fixed point at $x = 1/2$ is stable, and the other fixed points are unstable, the basin of attraction for the stable fixed point must be $]0, 1[$, as indicated. [1 mark]

(b) (i) For $\mu < 1/3$ the graph is thus



For $\mu > 1/3$ the graph is thus



[3 marks]
(ii) Fixed points are where $x = f(x)$.
Two cases to consider

$$x \leq 1/2$$

and

$$x > 1/2$$

If $x \leq 1/2$ we need

$$\begin{aligned} x = f(x) &= 3\mu x \\ \Rightarrow x(1-3\mu) &= 0 \\ \Rightarrow x &= 0 \qquad (\mu \neq 1/3) \end{aligned}$$

If $x > 1/2$ we need

$$\begin{aligned} x = f(x) &= 3\mu(1-x) \\ \Rightarrow x + 3\mu x &= 3\mu \\ \Rightarrow x &= \frac{3\mu}{1+3\mu} \end{aligned}$$

But $x = \dfrac{3\mu}{1+3\mu}$ is only $> 1/2$ as required if

$$\begin{aligned} \frac{3\mu}{1+3\mu} &> 1/2 \\ \Rightarrow 3\mu &> 1/2 + (3/2)\mu \\ \Rightarrow \mu &> 1/3. \end{aligned}$$

Hence, for $\mu \neq 1/3$, the fixed points are $x^* = 0$ for any μ and $x^* = \dfrac{3\mu}{1+3\mu}$ if $\mu > 1/3$. [7 marks]

(iii) For stability we need $|f'(x^*)| < 1$.

For $x^* = 0$, $f' = 3\mu$ so in this case we need

$$\begin{aligned} |3\mu| &< 1 \\ \Rightarrow 3\mu &< 1 \qquad (0 \le \mu \le 1) \\ \Rightarrow \mu &< 1/3 \end{aligned}$$

For $x^* = \dfrac{3\mu}{1+3\mu}$, $f' = -3\mu$.

But $\mu > 1/3$ for this fixed point so

$$|f'(x^*)| = |-3\mu| > 1$$

Hence this fixed point is unstable. [7 marks]

[All covered in problem sheets, tutorials and lectures]

2. (a) Consider the dynamical system obtained by iterating the map

$$f(x) = 1 - 2\mu x^2$$

for $x \in [-1, 1]$ and $0 < \mu < 2$.

Show that one of the fixed points of the system is at $x^* = (-1 + \sqrt{1 + 8\mu})/4\mu$ and show that this fixed point is stable if $\mu < 3/8$. [5 marks]

(b) Now investigate the properties of $f^{(2)}(x) = f(f(x))$.

(i) Show that this map has an additional fixed point at

$$x^* = \frac{1 + \sqrt{8\mu - 3}}{4\mu}.$$

[8 marks]

(ii) Show that x^* corresponds to a 2-cycle of $f(x)$ and that this is stable for $3/8 < \mu < 5/8$. [7 marks]

Solution

(a) For fixed points we need

$$\begin{aligned} x = f(x) &= 1 - 2\mu x^2 \\ \Rightarrow 2\mu x^2 + x - 1 &= 0 \\ \Rightarrow x &= \frac{-1 \pm \sqrt{1 + 8\mu}}{4\mu} \end{aligned}$$

Hence the system has a fixed point at

$$x^* = \frac{-1 + \sqrt{1 + 8\mu}}{4\mu}.$$

For stability we need $|f'(x^*)| < 1$.

Now $f'(x) = -4\mu x$ so

$$\begin{aligned} f'(x^*) &= -4\mu \left(\frac{-1 + \sqrt{1 + 8\mu}}{4\mu} \right) \\ &= 1 - \sqrt{1 + 8\mu} \end{aligned}$$

Hence for $|f'(x^*)| < 1$ we need

$$\begin{aligned}
-1 &< 1-\sqrt{1+8\mu}<1\\
\Rightarrow -2 &< -\sqrt{1+8\mu}<0\\
\Rightarrow 2 &> \sqrt{1+8\mu}>0\\
\Rightarrow 4 &> 1+8\mu>0 \qquad (\mu>0)\\
\Rightarrow 3 &> 8\mu>-1\\
\Rightarrow -1/8 &< \mu<3/8
\end{aligned}$$

Hence the fixed point is stable if $\mu < 3/8$ since $\mu > 0$. [5 marks]

(b) (i) For $f^{(2)}(x)$ we have

$$\begin{aligned}
f^{(2)}(x) &= 1-2\mu(1-2\mu x^2)^2\\
&= 1-2\mu(1+4\mu^2x^4-4\mu x^2)\\
&= 1-2\mu-8\mu^3x^4+8\mu^2x^2\\
&= -8\mu^3x^4+8\mu^2x^2-2\mu+1
\end{aligned}$$

Hence for fixed points we need

$$\begin{aligned}
x=f^{(2)}(x) &= -8\mu^3x^4+8\mu^2x^2-2\mu+1\\
\Rightarrow 8\mu^3x^4-8\mu^2x^2+x+2\mu-1 &= 0 \qquad (1)
\end{aligned}$$

Now we know that the fixed points of $f(x)$ are also fixed points of $f^{(2)}(x)$ so, from part (a), $(2\mu x^2+x-1)$ is a factor of the LHS of equation (1). Let the other factor be (Ax^2+Bx+C).

Substituting, expanding, and collecting terms we have

$$\begin{aligned}
(2\mu x^2+x-1)(Ax^2+Bx+C) &= 2\mu Ax^4+Ax^3-Ax^2+2\mu Bx^3+Bx^2-Bx\\
&+ 2\mu Cx^2+Cx-C\\
&= 2\mu Ax^4+(A+2\mu B)x^3+(-A+B+2\mu C)x^2\\
&+ (-B+C)x-C.
\end{aligned}$$

Comparing coefficients in this and the LHS of equation (1) we have:

For x^4

$$\begin{aligned}
2\mu A &= 8\mu^3\\
\Rightarrow A &= 4\mu^2.
\end{aligned}$$

For x^3

$$\begin{aligned} A + 2\mu B &= 0 \\ \Rightarrow B &= -\frac{A}{2\mu} = -2\mu. \end{aligned}$$

For constant terms

$$\begin{aligned} -C &= 2\mu - 1 \\ \Rightarrow C &= 1 - 2\mu. \end{aligned}$$

This is sufficient but we can also check with the x^2 and x terms.
For x^2

$$\begin{aligned} -A + B + 2\mu C &= -4\mu^2 - 2\mu + 2\mu - 4\mu^2 \\ &= -8\mu^2, \end{aligned}$$

which is correct.
For x

$$-B + C = 2\mu + 1 - 2\mu = 1,$$

which is correct.

Hence the two fixed points of $f^{(2)}(x)$ which are not also fixed points of $f(x)$, which we already know from above, are the roots of

$$4\mu^2 x^2 - 2\mu x + 1 - 2\mu = 0$$

so are

$$\begin{aligned} x^* &= \frac{2\mu \pm \sqrt{4\mu^2 - 16\mu^2(1-2\mu)}}{8\mu^2} \\ &= \frac{2\mu \pm \sqrt{32\mu^3 - 12\mu^2}}{8\mu^2} \\ &= \frac{2\mu \pm 2\mu\sqrt{8\mu - 3}}{8\mu^2} \\ &= \frac{1 \pm \sqrt{8\mu - 3}}{4\mu}. \end{aligned}$$

Hence $f^{(2)}(x)$ has an additional fixed point at

$$x^* = \frac{1+\sqrt{8\mu-3}}{4\mu}.$$

[8 marks]
(ii) Iterating using f starting at $x_0 = x^*$ we have

$$x_1 = f(x_0) = f(x^*).$$

Applying f again we have

$$\begin{aligned} x_2 &= f(x_1) \\ &= f(f(x_0)) \\ &= f(f(x^*)) \\ &= x^* \qquad \text{(since } x^* \text{ is a fixed point of } f(f(x))) \\ &= x_0. \end{aligned}$$

Hence x^* corresponds to a 2-cycle of $f(x)$.
For stability we need

$$\left| \frac{d(f(f(x)))}{dx} \right|_{x^*} < 1.$$

Using the chain rule we have

$$\left. \frac{d(f(f(x)))}{dx} \right|_{x^*} = \left. \frac{df}{dx} \right|_{x_1} \left. \frac{df}{dx} \right|_{x_2},$$

where x_1, x_2 are the two fixed points of $f^{(2)}$ which are not fixed points of f.
Hence we have

$$\begin{aligned} \left. \frac{d(f(f(x)))}{dx} \right|_{x^*} &= (-4\mu x_1)(-4\mu x_2) \\ &= 16\mu^2 \left(\frac{1+\sqrt{8\mu-3}}{4\mu} \right) \left(\frac{1-\sqrt{8\mu-3}}{4\mu} \right) \\ &= 1 - 8\mu + 3 \\ &= 4 - 8\mu. \end{aligned}$$

Hence for stability we need

$$\begin{aligned} |4-8\mu| &< 1 \\ \Rightarrow -1 &< 4-8\mu < 1 \\ \Rightarrow -5 &< -8\mu < -3 \\ \Rightarrow 5 &> 8\mu > 3 \\ \Rightarrow 3/8 &< \mu < 5/8. \end{aligned}$$

Hence the 2-cycle is stable for $3/8 < \mu < 5/8$, as required. [7 marks]
[All covered in problem sheets, tutorials and lectures]

3. Consider the dynamical systems defined by iterations of a function $f(x)$ in the following *four* cases:

$$\begin{array}{cll} \text{(i)} & f(x) = 3x \pmod 1 & \\ \text{(ii)} & f(x) = x + 2.1 \pmod 1 & \\ \text{(iii)} & f(x) = \sqrt{3}x^2, & 0 \leq x \leq 1 \\ \text{(iv)} & f(x) = 5x - 3x^3, & x \in \mathbb{R}. \end{array}$$

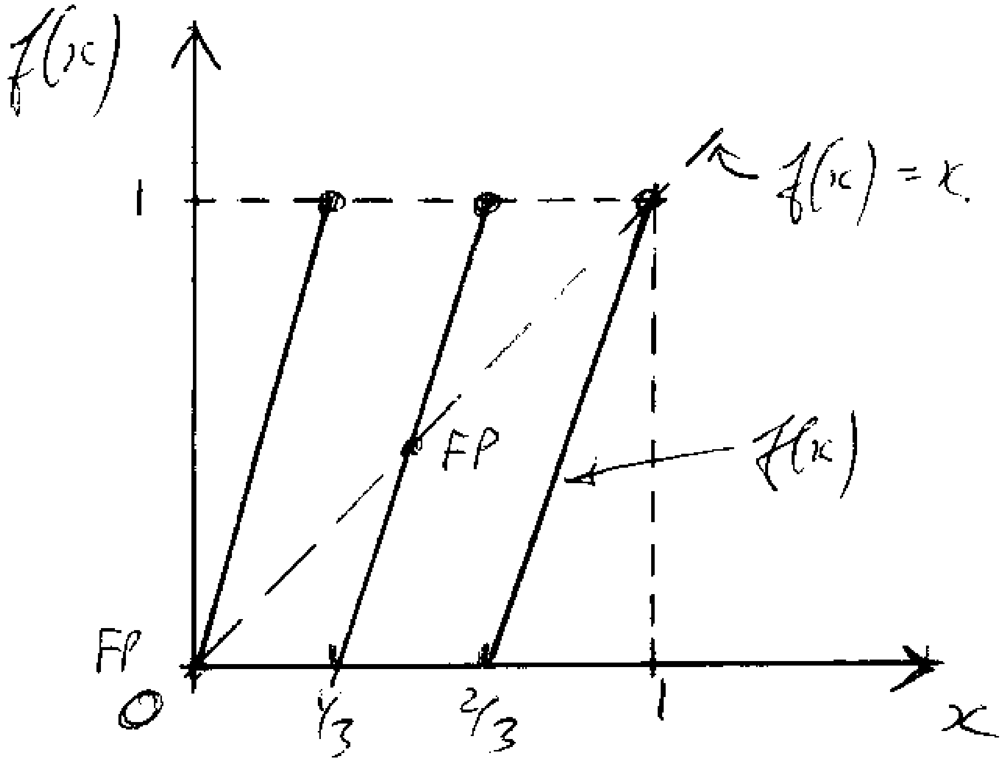
(a) In each case, find any fixed points and determine their stability. [8 marks]

(b) For cases (i) and (ii), find the Lyapunov exponent and say what you can deduce from its value. [8 marks]

(c) For cases (i) and (ii), discuss the limiting behaviour as $n \to \infty$ and how this is affected by the starting value, x_0. [4 marks]

Solution

(a) (i) Although the question does not ask for a graph, we do a sketch to help locate the fixed points. The graph is thus



From the graph we can see that there are two fixed points, which we need to find analytically, as follows.

For fixed points we need $x = f(x)$.

In this case we need

$$\begin{aligned} x &= f(x) = 3x \pmod 1 \\ \Rightarrow 3x &= x + 0 \Rightarrow x = 0 \end{aligned}$$

or

$$3x = x + 1 \Rightarrow x = 1/2.$$

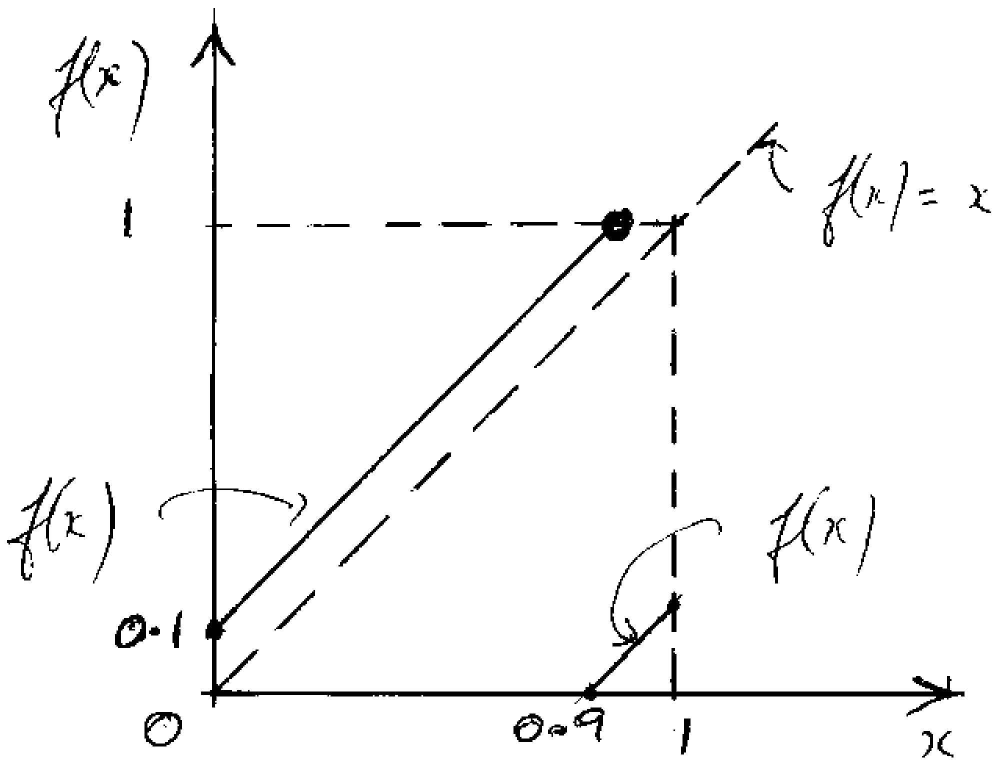
Note that $x = 1$ is not a fixed point since

$$\begin{aligned} f(1) &= 3 \pmod 1 \\ &= 0 \\ &\neq 1. \end{aligned}$$

Hence the fixed points are at $x = 0$ and $x = 1/2$.

Since $f'(x) = 3$ throughout the range of x, and $3 > 1$, both fixed points are unstable.

(ii) Again we do a graph, which is thus



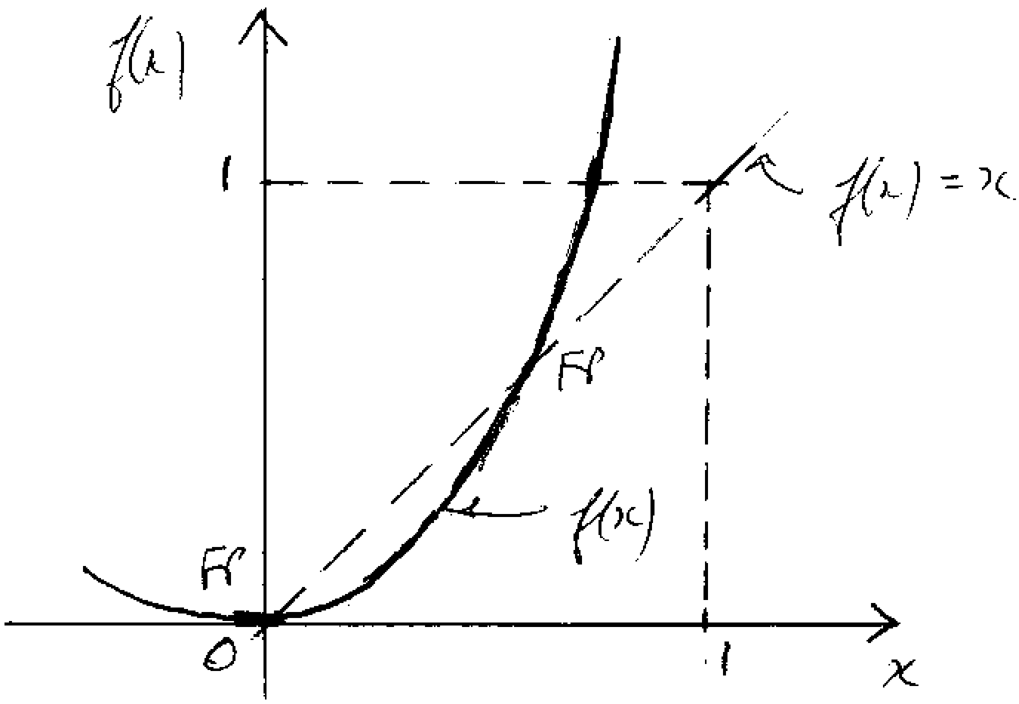
We see from the graph that there are no fixed points since the graph of $f(x)$ never intersects that of $f(x) = x$. We show this analytically as follows.

We need

$$\begin{aligned} x = f(x) &= x + 2.1 \qquad (\text{mod } 1) \\ \Rightarrow x + 2.1 &= x + 0 = x. \end{aligned}$$

This equation can never be true so there are no fixed points.

(iii) We note that the function is a simple quadratic and again we can draw a quick graph, which is thus



This shows that there are two fixed points, one at $x = 0$, but we need to locate the non-zero fixed point analytically.

We need

$$\begin{aligned} x = f(x) &= \sqrt{3}x^2 \\ \Rightarrow \sqrt{3}x^2 - x &= 0 \\ \Rightarrow x(\sqrt{3}x - 1) &= 0 \\ \Rightarrow x &= 0 \qquad \text{or} \qquad x = 1/\sqrt{3} \end{aligned}$$

Hence fixed points are at $x = 0$ and $x = 1/\sqrt{3}$

We have $f'(x) = 2\sqrt{3}x$

Hence at $x = 0$, $f'(0) = 0$ so the fixed point at $x = 0$ is superstable.

At $x = 1/\sqrt{3}$, $f'(1/\sqrt{3}) = 2$ so $|f'| > 1$ and the fixed point is unstable.

(iv) The function is too complicated for it to be worth doing a graph. Hence we work purely analytically.

In this case we need

$$x = f(x) = 5x - 3x^3$$

$$\begin{aligned}
\Rightarrow 3x^3 - 4x &= 0 \\
\Rightarrow x(3x^2 - 4) &= 0 \\
\Rightarrow x &= 0 \qquad \text{or} \qquad 3x^2 = 4 \Rightarrow x = \pm 2/\sqrt{3}
\end{aligned}$$

Hence fixed points are at $x = 0$, $x = 2/\sqrt{3}$ and $x = -2/\sqrt{3}$.

$f'(x) = 5 - 9x^2$.

Hence at $x = 0$, $f'(0) = 5 > 1$ so the fixed point is unstable.

At both $x = 2/\sqrt{3}$ and at $x = -2/\sqrt{3}$, we have $f' = 5 - 9 \times 4/3 = -7$ so $|f'| > 1$ and both fixed points are unstable. [8 marks]

(b) (i) Starting with x_0, the iterations are as follows

$$\begin{aligned}
x_1 &= 3x_0 \qquad (\text{mod } 1) \\
x_2 &= 3x_1 = 3^2 x_0 \qquad (\text{mod } 1) \\
&\cdots \\
&\cdots \\
x_n &= 3^n x_0 \qquad (\text{mod } 1)
\end{aligned}$$

Hence $\dfrac{dx_n}{dx_0} = 3^n$ so the Lyapunov exponent, λ, is given by

$$\begin{aligned}
\lambda &= \lim_{n\to\infty} \frac{1}{n} \ln \left| \frac{dx_n}{dx_0} \right| \\
&= \lim_{n\to\infty} \frac{1}{n} \ln(3^n) \\
&= \lim_{n\to\infty} \frac{1}{n} n \ln(3) = \ln(3) > 0
\end{aligned}$$

Since $\lambda > 0$ the motion can be chaotic (depending on the starting value), with nearby trajectories diverging.

(b) (ii) Starting with x_0, the iterations are as follows

$$\begin{aligned}
x_1 &= x_0 + 2.1 \qquad (\text{mod } 1) \\
x_2 &= x_1 + 2.1 = x_0 + 2 \times 2.1 \qquad (\text{mod } 1) \\
&\cdots \\
&\cdots \\
x_n &= x_0 + n \times 2.1 \qquad (\text{mod } 1)
\end{aligned}$$

Hence $\dfrac{dx_n}{dx_0} = 1$ so the Lyapunov exponent, λ, is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln 1 = 0$$

Hence, the motion is not chaotic and, since 2.1 is a rational number, the motion is periodic. [8 marks]

(c) (i) From above, the fixed points are unstable and the motion chaotic if not starting at a fixed point and if x_0 is irrational. If x_0 is rational we have periodic behaviour (including 1cycles). There is sensitive dependence on the initial conditions, with neighbouring trajectories diverging.

(ii) Since 2.1 is rational, the motion is periodic. Since, in its lowest terms, we have $2.1 = \dfrac{21}{10}$, we have a 10-cycle. We can show this as follows.

$$\begin{aligned} x_{n+10} &= x_n + 10 \times \frac{21}{10} \qquad (\text{mod } 1) \\ &= x_n + 21 \qquad (\text{mod } 1) \\ &= x_n \end{aligned}$$

[4 marks]

[Past exam question. All covered in tutorials, lectures and classwork]

4. Consider the dynamical system $x_{n+1} = f(x_n, y_n)$, $y_{n+1} = g(x_n, y_n)$ generated by the functions

$$\begin{aligned} f(x,y) &= x^2 - y^2 + a \\ g(x,y) &= 3xy, \end{aligned}$$

where a is a constant.

(i) Show that the system has fixed points given by $x^* = \frac{1}{2}\left(1 \pm \sqrt{1-4a}\right)$, $y^* = 0$ for $a < 1/4$ and $x^* = 1/3$, $y^* = \pm\frac{1}{3}\left(\sqrt{9a-2}\right)$ for $a > 2/9$. [7 marks]

(ii) Linearize the system about the appropriate fixed points for $a < 2/9$ and show that the system has a stable fixed point for $-4/9 < a < 2/9$. [8 marks]

(iii) Consider the set of points on a circle of radius r centred at the origin. Show that they are mapped under one step of this dynamical system to an ellipse and sketch the ellipse for $a = 2, r = 1$. [5 marks]

Solution

(i) The system is

$$\begin{aligned} f(x,y) &= x^2 - y^2 + a \\ g(x,y) &= 3xy \end{aligned}$$

where a is a constant.

For the fixed points we need, simultaneously

$$x = f(x,y) = x^2 - y^3 + a \qquad (1)$$

and

$$y = g(x,y) = 3xy \qquad (2).$$

Rearranging equation (2) we have

$$\begin{aligned} y(3x-1) &= 0 \\ \Rightarrow y &= 0 \qquad \text{or} \qquad x = 1/3. \end{aligned}$$

For $y = 0$, substituting in equation (1) gives

$$\begin{aligned} x^2 - x + a &= 0 \\ \Rightarrow x &= \frac{1 \pm \sqrt{1-4a}}{2} \end{aligned}$$

This is an acceptable (real) solution for $1 - 4a > 0 \Rightarrow a < 1/4$.
Hence there are fixed points at

$$x^* = \frac{1}{2}(1 \pm \sqrt{1-4a}), y^* = 0 \qquad (a < 1/4).$$

For $x = 1/3$, substituting in equation (1) gives

$$\begin{aligned} 1/3 &= 1/9 - y^2 + a \\ \Rightarrow y^2 &= a - 2/9 \\ \Rightarrow y &= \pm\frac{1}{3}(\sqrt{9a-2}). \end{aligned}$$

This is an acceptable (real) solution for $9a - 2 > 0 \Rightarrow a > 2/9$.
Hence the other fixed points are at

$$x^* = 1/3, y^* = \pm\frac{1}{3}(\sqrt{9a-2}) \qquad (a > 2/9).$$

[7 marks]
(ii) For $a < 2/9$ we only need to consider the first fixed points.
The Jacobian matrix is given by

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ 3y & 3x \end{bmatrix}.$$

Since $y = 0$ for both fixed points, it is easiest to substitute this value in the Jacobian matrix to give, at the fixed points,

$$J|_{FP's} = \begin{bmatrix} 2x & 0 \\ 0 & 3x \end{bmatrix}.$$

We note that this is a diagonal matrix so its eigenvalues are just the diagonal entries so, denoting the fixed points as x^*_+ and x^*_- in the usual way, and the two eigenvalues in each case by λ_1 and λ_2 we have

For x^*_+

$$\lambda_1 = 2x_+^* = 1 + \sqrt{1-4a}$$

and

$$\lambda_2 = 3x_+^* = \frac{3}{2}(1 + \sqrt{1-4a}).$$

Hence, since $\sqrt{1-4a} > 0$ for $a < 2/9$, $|\lambda_1|, |\lambda_2| > 1$ so this fixed point is unstable.

For x_-^*

$$\lambda_1 = 2x_-^* = 1 - \sqrt{1-4a}$$

and

$$\lambda_2 = 3x_-^* = \frac{3}{2}(1 - \sqrt{1-4a}).$$

$|\lambda_1|$ is clearly < 1 for $a < 2/9$.

For $|\lambda_2| < 1$ we need

$$\begin{aligned}
|\frac{3}{2}(1 - \sqrt{1-4a})| &< 1 \\
\Rightarrow -1 < \frac{3}{2}(1 - \sqrt{1-4a}) &< 1 \\
\Rightarrow -2/3 < 1 - \sqrt{1-4a} &< 2/3 \\
\Rightarrow -5/3 < -\sqrt{1-4a} &< -1/3 \\
\Rightarrow 5/3 > \sqrt{1-4a} &> 1/3 \\
\Rightarrow 25/9 > 1 - 4a &> 1/9 \\
\Rightarrow 16/9 > -4a &> -8/9 \\
\Rightarrow -16/9 < 4a &< 8/9 \\
\Rightarrow -4/9 < a &< 2/9
\end{aligned}$$

Hence there is a stable fixed point for $-4/9 < a < 2/9$. [8 marks]

(iii) Points on a circle of radius r centred at the origin have coordinates

$$x_0 = r\cos\theta, \qquad y_0 = r\sin\theta,$$

where r and θ are the usual polar coordinates.

Substituting in the system equations, under one step of the system the circle is mapped to the set of points (x_1, y_1) given by

$$\begin{aligned} x_1 = x_0^2 - y_0^2 + a &= r^2\cos^2\theta - r^2\sin^2\theta + a \\ &= r^2\cos(2\theta) + a \end{aligned}$$

and

$$\begin{aligned} y_1 = 3x_0y_0 &= 3r^2\sin\theta\cos\theta \\ &= \frac{3}{2}r^2\sin(2\theta) \end{aligned}$$

Changing variables to $x_1 - a = z_1$ to move the origin to the point $(a, 0)$ we have

$$\begin{aligned} x_1 - a = z_1 &= r^2\cos(2\theta) \\ y_1 &= \frac{3}{2}r^2\sin(2\theta). \end{aligned}$$

Hence

$$\cos(2\theta) = \frac{z_1}{r^2} \qquad \text{and} \qquad \sin(2\theta) = \frac{2}{3}\frac{y_1}{r^2}$$

so, since $\cos^2(2\theta) + \sin^2(2\theta) = 1$,

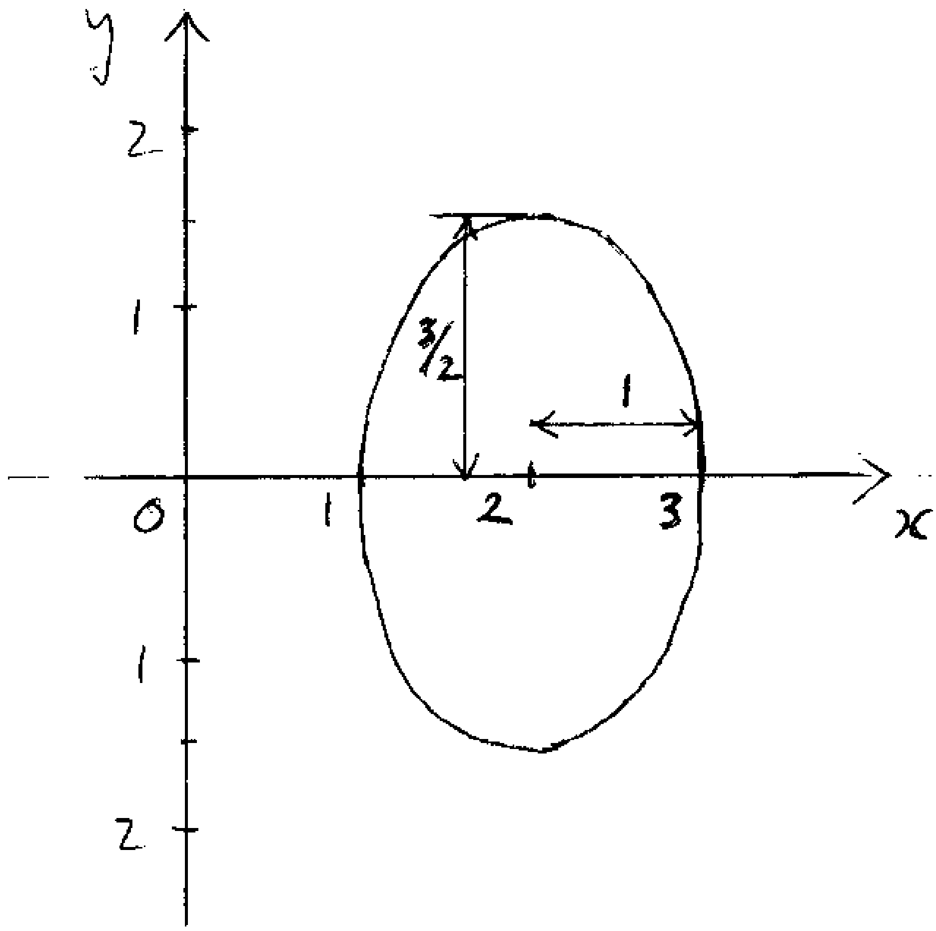
$$\begin{aligned} \frac{z_1^2}{r^4} + \frac{4y_1^2}{9r^4} &= 1 \\ \Rightarrow \frac{(x_1-a)^2}{r^4} + \frac{y_1^2}{\frac{9}{4}r^4} &= 1. \end{aligned}$$

Comparing with the standard equation

$$\frac{x^2}{A^2} + \frac{y_2}{B^2} = 1$$

for an ellipse centred at the origin, with semi-axes of lengths A and B, this is an ellipse centred at $(a, 0)$ with semi-axes of lengths r^2 and $\frac{3}{2}r^2$.

With $a = 2$ and $r = 1$ the ellipse is as shown below.



[5 marks]
[All covered in problem sheets, tutorials and lectures]

5. Consider the dynamical system described by the equations

$$\begin{aligned} \frac{dx}{dt} &= (1-x)(1-bx) + 2x^2y \\ \frac{dy}{dt} &= bx(1-x) - 2x^2y, \end{aligned}$$

where b is a real positive parameter.

(i) Find the fixed point of the system and discuss its stability for $b > 0$, with $b \neq 3$ and $b \neq 3 + 2\sqrt{2}$. [8 marks]

(ii) For the particular case of the system when $b = 4$, consider trajectories which pass through the four points $(1/4, 0)$, $(1, 1)$, $(2, 0)$, and $(1, -1)$ and sketch the directions of the tangents to these trajectories. [7 marks]

(iii) Give *plausibility only* arguments that the system has a stable limit cycle when $b \approx 4$. [3 marks]

(iv) Discuss whether this system exhibits chaotic behaviour. [2 marks]

Solution

(i) For the fixed point we need, simultaneously,

$$\frac{dx}{dt} = f(x, y) = (1-x)(1-bx) + 2x^2y = 0 \qquad (1)$$

and

$$\frac{dy}{dt} = g(x, y) = bx(1-x) - 2x^2y = 0 \qquad (2)$$

Noting the common term $2x^2y$ we add equations (1) and (2) to give

$$\begin{aligned} (1-x)(1-bx) + bx(1-x) &= 0 \\ \Rightarrow (1-x)(1-bx+bx) &= (1-x) = 0 \\ \Rightarrow x &= 1. \end{aligned}$$

Substituting in equation (2) now gives

$$\begin{aligned} -2y &= 0 \\ \Rightarrow y &= 0. \end{aligned}$$

Hence the system has a single fixed point at $(1, 0)$.

To investigate the stability of the fixed point we need the Jacobian matrix

$$J = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix}$$

evaluated at $(1, 0)$.
Differentiating, we have

$$\begin{aligned} \frac{\partial f}{\partial x} &= -(1 - bx) - b(1 - x) + 4xy \\ &= -1 - b + 2bx + 4xy \end{aligned}$$

so $\left.\dfrac{\partial f}{\partial x}\right|_{(1,0)} = b - 1.$

$$\frac{\partial f}{\partial y} = 2x^2$$

so $\left.\dfrac{\partial f}{\partial y}\right|_{(1,0)} = 2.$

$$\frac{\partial g}{\partial x} = b(1 - x) - bx - 4xy$$

so $\left.\dfrac{\partial g}{\partial x}\right|_{(1,0)} = -b.$

$$\frac{\partial g}{\partial y} = -2x^2$$

so $\left.\dfrac{\partial g}{\partial y}\right|_{(1,0)} = -2.$
Hence we have

$$J|_{(1,0)} = \begin{bmatrix} b - 1 & 2 \\ -b & -2 \end{bmatrix}.$$

Characteristic equation to find eigenvalues of J is

$$\begin{aligned} \det(J - \lambda I) &= 0 \\ \Rightarrow \begin{vmatrix} b - 1 - \lambda & 2 \\ -b & -2 - \lambda \end{vmatrix} &= 0 \\ \Rightarrow (b - 1 - \lambda)(-2 - \lambda) + 2b &= 0 \end{aligned}$$

$$\begin{aligned}
\Rightarrow -2b+2+2\lambda-b\lambda+\lambda+\lambda^2+2b &= 0\\
\Rightarrow \lambda^2+(3-b)\lambda+2 &= 0\\
\Rightarrow \lambda &= \frac{(b-3)\pm\sqrt{(b-3)^2-8}}{2}.
\end{aligned}$$

The nature of λ depends on the square root term. If

$$\begin{aligned}
(b-3)^2-8 &< 0\\
\Rightarrow (b-3)^2 &< 8\\
\Rightarrow -2\sqrt{2} &< b-3<2\sqrt{2}\\
3-2\sqrt{2} &< b<3+2\sqrt{2}
\end{aligned}$$

we have a complex conjugate pair of eigenvalues

$$\lambda_\pm=\frac{(b-3)\pm i\sqrt{8-(b-3)^2}}{2}$$

and, hence, spiral behaviour.

Stability depends on $\mathrm{Re}(\lambda_\pm)$.

If $b>3$, $\mathrm{Re}(\lambda_\pm)>0$ so the fixed point is an unstable spiral repellor.

If $b<3$, $\mathrm{Re}(\lambda_\pm)<0$ so the fixed point is a stable spiral attracting node.

$b\neq 3$ given.

If $(b-3)^2\geq 8\Rightarrow 0<b\leq 3-2\sqrt{2}$ (≈ 0.17) or $b>3+2\sqrt{2}$ (≈ 5.83) there are two real eigenvalues.

If $0<b\leq 3-2\sqrt{2}$ we have $-3<b-3\leq -2\sqrt{2}$ so $b-3<0$ and hence $3-b>0$. λ_- is clearly <0 in this case. λ_+ is also <0 since we have

$$\begin{aligned}
\sqrt{(b-3)^2-8}=\sqrt{(3-b)^2-8} &< 3-b \qquad (3-b>0)\\
\Rightarrow b-3+\sqrt{(3-b)^2-8} &< 0
\end{aligned}$$

Hence in this case the FP is a stable node.

If $b>3+2\sqrt{2}$, on the other hand, $b-3>0$ so $\sqrt{(b-3)^2-8}<b-3$ and hence $\lambda_\pm>0$.

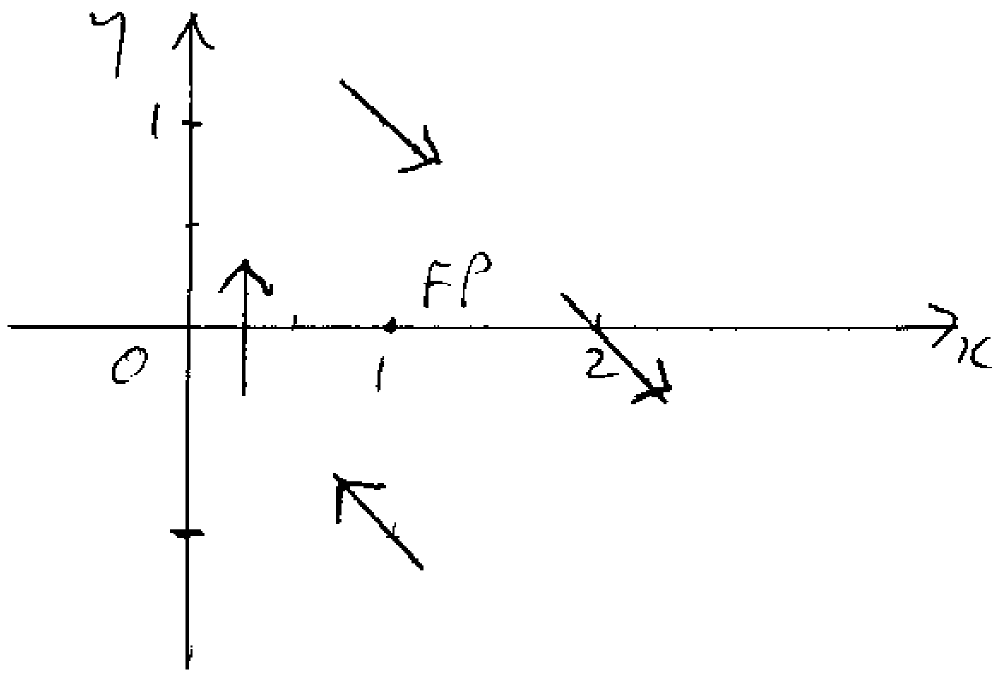
Hence in this case the FP is an unstable node. [8 marks]

(ii) Case b=4.

Tabulating the values at the given points we have

(x,y)	dx/dt	dy/dt	$dy/dx = \frac{dy/dt}{dx/dt}$
	$(1-x)(1-4x)+2x^2y$	$4x(1-x)-2x^2y$	
$(1/4,0)$	0	>0	$+\infty$
$(1,1)$	2	-2	-1
$(2,0)$	7	-8	$-8/7$
$(1,-1)$	-2	2	-1

Hence the tangent directions are as shown below.



[7 marks]

(iii) From part (i), the fixed point at $(1,0)$ is an unstable spiral repellor for $b=4$. From the tangent directions, the trajectory is close to a limit cycle. This suggests that there is a limit cycle when $b \approx 4$.

Now $b=4$ is between the change from a spiral attractor to a spiral repellor at $b=3$ and the change from a spiral repellor to an unstable node at $b=3+2\sqrt{2}$. It is plausible that there is a Hopf bifurcation at $b=3$, giving rise to the spiral repellor with a surrounding limit cycle. [3 marks]

(iv) Being a 2-D continuous time system with ODE's, this system cannot exhibit chaotic behaviour because of the Poincare-Bendixson Theorem, which states that trajectories approach either a fixed point or a limit cycle. 3-D is required for chaos. The Poincare-Bendixson Theorem is a result of the region being bounded and the No-crossing Theorem. [2 marks]

[All covered in problem sheets, tutorials and lectures]

6. (a) The Sierpinski carpet is constructed from a unit square by dividing the square into 3×3 smaller equal squares, removing the central smaller square to leave the 8 smaller squares around the perimeter, then repeating the procedure for these 8 squares, and so on.

(i) Sketch the first three levels of this process, starting with and including the unit square itself. [2 marks]

(ii) Find the capacity dimension of the resulting infinite set. [4 marks]

(b) A dynamical system on $[0, 1]$ is given by

$$x_{n+1} = f(x_n)$$

where

$$\begin{aligned} f(x) &= 0 \qquad \text{for} \quad \frac{1}{4} < x < \frac{3}{4} \\ f(x) &= 4x \qquad (\text{mod } 1), \qquad \text{otherwise.} \end{aligned}$$

(i) Sketch the graph of $f(x)$. [2 marks]

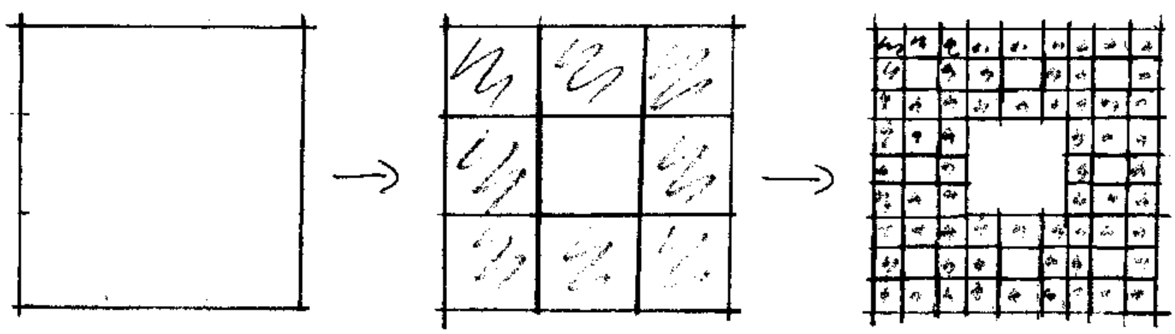
(ii) Show that the fixed points of this system are unstable. [2 marks]

(iii) Consider the set S of initial points x_0 for which $x_n \neq 0$ as $n \to \infty$. Obtain a description of S and use it to find the capacity dimension of S. [8 marks]

(iv) Give an example in base 4 of an initial value x_0 for which the system will show periodic behaviour. [2 marks]

Solution

(a) (i) The first three stages are thus



[2 marks]

(ii) Using boxes which cover the individual small squares, and tabulating the levels, we have

Level	*No. of boxes, $N(\epsilon)$*	*Size of boxes, ϵ*
0	1	1
1	8	$1/3$
2	$8 \times 8 = 8^2$	$1/3 \times 1/3 = (1/3)^2$
$\vdots$	$\vdots$	$\vdots$
n	8^n	$(1/3)^n$

Hence, for the capacity dimension, d_c, we need the limit, as $n \to \infty$, of

$$N(\epsilon) = 8^n = A((1/3)^n)^{-d_c} = A(3)^{nd_c}$$

where A is a constant.
Taking logs we have

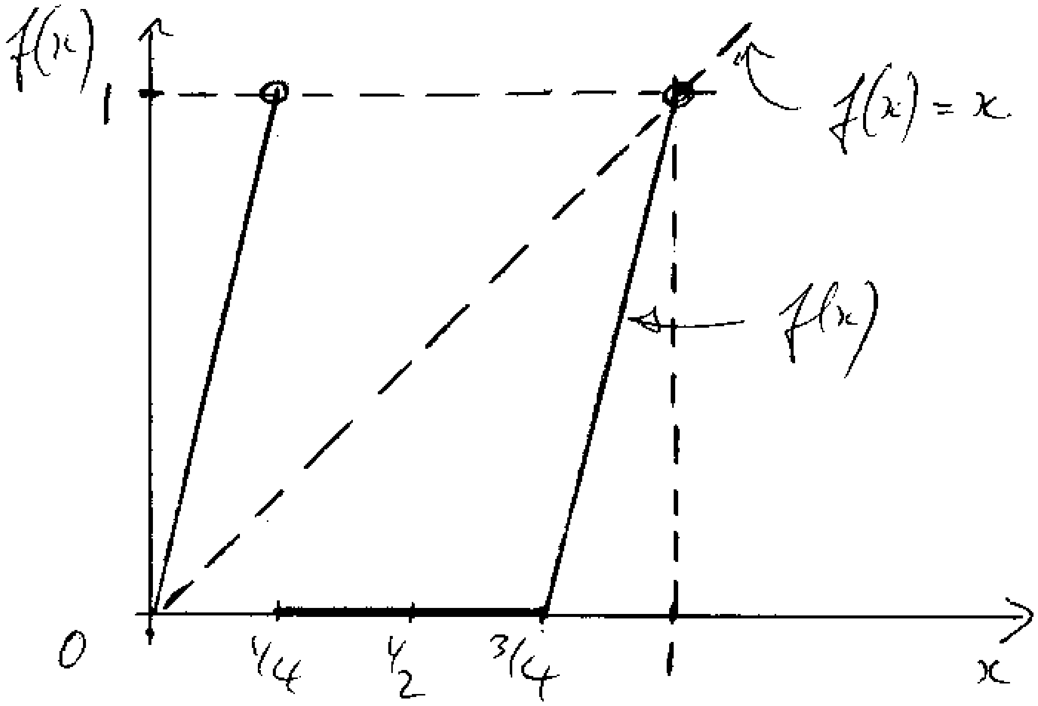
$$\begin{aligned} n \ln 8 &= \ln A + nd_c \ln 3 \\ \Rightarrow \ln 8 &= \frac{1}{n} \ln A + d_c \ln 3. \end{aligned}$$

The first term on the RHS clearly $\to 0$ as $n \to \infty$ so we have

$$\begin{aligned} \ln 8 &= d_c \ln 3 \\ \Rightarrow d_c &= \frac{\ln 8}{\ln 3} \approx 1.893. \end{aligned}$$

[4 marks]

(b) (i) The graph of $f(x)$ is thus



[2 marks]

(ii) From the graph, there is only one fixed point, at $x = 0$.

The point $x = 1$ is not a fixed point since $f(1) = 3 \pmod 1 = 0 \neq 1$

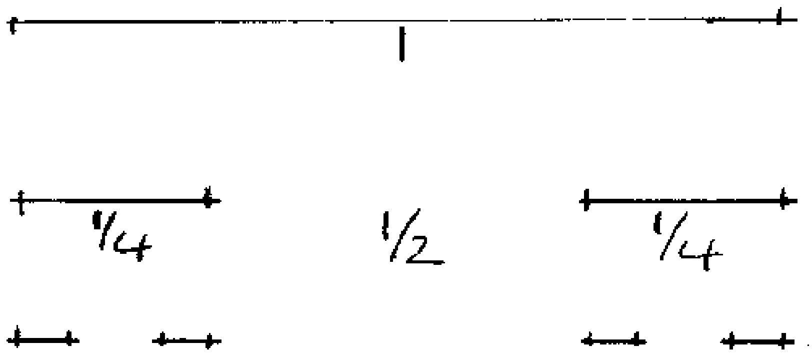
At $x = 0$, $f'(x) = 4$ so $|f'(0)| > 1$ and the fixed point is unstable. [2 marks]

(iii) The set S of initial points for which $x_n \neq 0$ as $n \to \infty$ can be described as follows.

Divide the interval $[0, 1]$ into four equal subintervals, as in the above graph, namely the $1^{st}, 2^{nd}, 3^{rd}$ and 4^{th} subintervals.

Denote by Q the operation of removal of the 2^{nd} and 3^{rd} subintervals, leaving the other two. Now apply Q to each of the two remaining subintervals. Clearly, $f(f(x)) = 0$ for every x in the 2^{nd} and 3^{rd} parts of these subintervals, leaving us with $2^2 = 4$ subintervals. This process is repeated indefinitely to remove all x_0 for which $x_n \to 0$ as $n \to \infty$.

Graphically, the procedure is similar to that for the Cantor set, but removing the section of length 1/2 of the segment length in the centre of each of the segments at each stage, thus



Using boxes to cover the line segments and tabulating the results we have

Level	*No. of boxes,* $N(\epsilon)$	*Size of boxes,* ϵ
0	1	1
1	2	1/4
2	$2 \times 2 = 2^2$	$1/4 \times 1/4 = (1/4)^2$
$\vdots$	$\vdots$	$\vdots$
n	2^n	$(1/4)^n$

Using the same procedure as in part (a)(ii) we have

$$N(\epsilon) = 2^n = A((1/4)^n)^{-d_c} = A(4)^{nd_c}.$$

Taking logs we have

$$n \ln 2 = \ln A + nd_c \ln 4$$

so, in the limit as $n \to \infty$,

$$d_c = \frac{\ln 2}{\ln 4} = 0.5.$$

[8 marks]

(iv) A recurring expression with no "1" or "2" digits in base 4 will correspond to a cycle and hence show periodic behaviour. For example, in base 4, the number

$$0.003003003 \ldots\ldots \text{recurring}$$

corresponds to a 3-cycle. [2 marks]

[All covered in problem sheets, tutorials and lectures. Also past exam question and classwork]

7. Consider the Lorenz system

$$\begin{aligned}\frac{dx}{dt} &= y - x\\ \frac{dy}{dt} &= \rho x - y - xz\\ \frac{dz}{dt} &= -z + xy\ ,\end{aligned}$$

with ρ a real positive constant.

(i) Show that the origin is a fixed point, $P_1 = (0, 0, 0)$, and that its stability depends on eigenvalues λ satisfying

$$(\lambda + 1)\left[\lambda^2 + 2\lambda + 1 - \rho\right] = 0\ .$$

[5 marks]

(ii) Deduce that this fixed point is stable only when $0 < \rho < 1$. [5 marks]

(iii) Show that there are two further fixed points

$$P_2\ ,\ P_3 = \left\{\left(\pm(\rho - 1)^{1/2}\ ,\ \pm(\rho - 1)^{1/2}\ ,\ (\rho - 1)\right)\right\}\ ,$$

when $\rho > 1$ and that their stability depends on eigenvalues λ satisfying

$$\lambda^3 + 3\lambda^2 + (1 + \rho)\lambda + 2(\rho - 1) = 0\ .$$

[7 marks]

(iv) Show that, if $\bar{z} = z - \rho - 1$, then

$$\frac{1}{2}\frac{d}{dt}\left(x^2 + y^2 + \bar{z}^2\right) = -x^2 - y^2 - \left[\bar{z} + \frac{1}{2}(\rho + 1)\right]^2 + \frac{1}{4}(\rho + 1)^2\ ,$$

so that $\sqrt{x^2 + y^2 + \bar{z}^2}$ decreases for all states outside a particular sphere (implying the existence of an attractor). [3 marks]

Solution

(i) The system is

$$\begin{aligned}\frac{dx}{dt} &= y - x = f(x, y, z)\\ \frac{dy}{dt} &= \rho x - y - xz = g(x, y, z)\\ \frac{dz}{dt} &= -z + xy = h(x, y, z)\end{aligned}$$

with $\rho > 0$.

For a fixed point we need $\dfrac{dx}{dt} = \dfrac{dy}{dt} = \dfrac{dz}{dt} = 0$.

This is clearly true for $x = y = z = 0$ so $P_1 = (0,0,0)$ is a fixed point, as required.

To investigate stability we need the eigenvalues of the Jacobian matrix, J, given by

$$J = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} & \dfrac{\partial h}{\partial z} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ \rho - z & -1 & -x \\ y & x & -1 \end{bmatrix}.$$

Substituting $(x,y,z) = (0,0,0)$ we have

$$J|_{P_1} = \begin{bmatrix} -1 & 1 & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

For the eigenvalues we need to solve $\det(J - \lambda I) = 0$, giving

$$\begin{aligned} \begin{vmatrix} -1-\lambda & 1 & 0 \\ \rho & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} &= 0 \\ \Rightarrow (-1-\lambda)(-1-\lambda)(-1-\lambda) - (\rho(-1-\lambda)) &= 0 \\ \Rightarrow -(1+2\lambda+\lambda^2)(1+\lambda) + \rho(1+\lambda) &= 0 \\ \Rightarrow (1+\lambda)[\lambda^2 + 2\lambda + 1 - \rho] &= 0 \qquad (1) \end{aligned}$$

as required. [5 marks]

(ii) For stability we need all solutions of equation (1) to have $\lambda < 0$, or to have $\mathrm{Re}(\lambda) < 0$.

$\lambda = -1 < 0$ is one obvious solution.

Using the formula for the other two, and denoting them by $\lambda_\pm$, we have

$$\begin{aligned} \lambda_\pm &= \frac{-2 \pm \sqrt{4 - 4(1-\rho)}}{2} \\ &= -1 \pm \sqrt{\rho} \end{aligned}$$

Clearly $\lambda_- < 0$ since $\sqrt{\rho} > 0$ by definition.

For $\lambda_+ < 0$ we need

$$\begin{aligned} -1+\sqrt{\rho} &< 0 \\ \Rightarrow \sqrt{\rho} &< 1 \\ \Rightarrow \rho &< 1 \end{aligned}$$

Hence, since $\rho > 0$ by definition, P_1 is stable only when $0 < \rho < 1$. [5 marks]
(iii) Fixed points are where

$$\begin{aligned} \frac{dx}{dt} &= y - x = 0 \Rightarrow y = x \qquad (1) \\ \frac{dy}{dt} &= \rho x - y - xz = 0 \qquad (2) \\ \frac{dz}{dt} &= -z + xy = 0 \qquad (3) \end{aligned}$$

Substituting from equation (1) in equation (2) we have

$$\begin{aligned} \rho x - x - xz &= 0 \\ \Rightarrow x(\rho - 1 - z) &= 0 \\ \Rightarrow x &= 0 \qquad \text{or} \qquad z = \rho - 1. \end{aligned}$$

Substituting $y = x$ and $z = \rho - 1$ in equation (3) we have

$$\begin{aligned} -\rho + 1 + x^2 &= 0 \\ \Rightarrow x &= \pm\sqrt{\rho - 1} \qquad (\rho > 1 \text{ so that } x \text{ is real}) \\ \Rightarrow y &= x = \pm\sqrt{\rho - 1} \end{aligned}$$

Hence there are two further fixed points at

$$P_2, P_3 = \{(\pm(\rho-1)^{1/2}, \pm(\rho-1)^{1/2}, (\rho-1)\} \qquad (\rho > 1).$$

Let $R = \pm(\rho - 1)^{1/2}$.
Substituting in the Jacobian matrix as before we have

$$J|_{P_2,P_3} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -R \\ R & R & -1 \end{bmatrix}$$

Hence, using $\det(J-\lambda I)=0$, we have

$$\left|\begin{array}{lcr} -1-\lambda & 1 & 0 \\ 1 & -1-\lambda & -R \\ R & R & -1-\lambda \end{array}\right|_{P_2,P_3} = 0$$

$$\begin{aligned}
\Rightarrow (-1-\lambda)((-1-\lambda)(-1-\lambda)+R^2)-((-1-\lambda)+R^2) &= 0\\
\Rightarrow (-1-\lambda)(1+2\lambda+\lambda^2+R^2)+1+\lambda-R^2 &= 0\\
\Rightarrow -1-2\lambda-\lambda^2-R^2-\lambda-2\lambda^2-\lambda^3-R^2\lambda+1+\lambda-R^2 &= 0\\
\Rightarrow \lambda^3+3\lambda^2+(2+R^2)\lambda+2R^2 &= 0\\
\Rightarrow \lambda^3+3\lambda^2+(2+\rho-1)\lambda+2(\rho-1) &= 0\\
\Rightarrow \lambda^3+3\lambda^2+(1+\rho)\lambda+2(\rho-1) &= 0
\end{aligned}$$

as required. [7 marks]
(iv) Let $\overline{z}=z-\rho-1$.
Using the chain rule we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(x^2+y^2+\overline{z}^2) &= \frac{1}{2}(2x\frac{dx}{dt}+2y\frac{dy}{dt}+2\overline{z}\frac{d\overline{z}}{dt})\\
&= x\frac{dx}{dt}+y\frac{dy}{dt}+\overline{z}\frac{d\overline{z}}{dt}.
\end{aligned}$$

Now from the definition of $\overline{z}$, we have $\dfrac{d\overline{z}}{dt}=\dfrac{dz}{dt}$ so, substituting using the system equations for $\dfrac{dx}{dt}$ etc. we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(x^2+y^2+\overline{z}^2) &= x\frac{dx}{dt}+y\frac{dy}{dt}+\overline{z}\frac{d\overline{z}}{dt}\\
&= x(y-x)+y(\rho x-y-x(\overline{z}+\rho+1))+\overline{z}(-\overline{z}-\rho-1+xy)\\
&= xy-x^2+\rho xy-y^2-xy\overline{z}-xy\rho-xy-\overline{z}^2-\overline{z}\rho-\overline{z}+xy\overline{z}\\
&= -x^2-y^2-\overline{z}^2-\overline{z}(\rho+1)\\
&= -x^2-y^2-\left[\overline{z}+\frac{1}{2}(\rho+1)\right]^2+\frac{1}{4}(\rho+1)^2
\end{aligned}$$

by completing the square, as required.
For $\sqrt{x^2+y^2+\overline{z}^2}$ to decrease we need

$$\frac{d}{dt}(x^2+y^2+\overline{z}^2)<0,$$

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which is true provided that

$$
\begin{aligned}
-x^2 - y^2 - \left[\overline{z} + \frac{1}{2}(\rho+1)\right]^2 + \frac{1}{4}(\rho+1)^2 &< 0 \\
\Rightarrow \frac{-x^2 - y^2 - \left[\overline{z} + \frac{1}{2}(\rho+1)\right]^2 + \frac{1}{4}(\rho+1)^2}{\frac{1}{4}(\rho+1)^2} &< 0 \\
\Rightarrow \frac{-x^2 - y^2 - \left[\overline{z} + \frac{1}{2}(\rho+1)\right]^2}{\frac{1}{4}(\rho+1)^2} + 1 &< 0 \\
\Rightarrow \frac{x^2 + y^2 + \left[\overline{z} + \frac{1}{2}(\rho+1)\right]^2}{\frac{1}{4}(\rho+1)^2} &> 1 \\
\Rightarrow \frac{x^2}{\frac{1}{4}(\rho+1)^2} + \frac{y^2}{\frac{1}{4}(\rho+1)^2} + \frac{\left[\overline{z} + \frac{1}{2}(\rho+1)\right]^2}{\frac{1}{4}(\rho+1)^2} &> 1.
\end{aligned}
$$

Comparing the LHS of this equation with the standard form for a sphere of

$$\frac{x^2}{A^2} + \frac{y^2}{A^2} + \frac{z^2}{A^2} = 1$$

shows that $\sqrt{x^2 + y^2 + \overline{z}^2}$ decreases outside the sphere with equation

$$\frac{x^2}{\frac{1}{4}(\rho+1)^2} + \frac{y^2}{\frac{1}{4}(\rho+1)^2} + \frac{\left[\overline{z} + \frac{1}{2}(\rho+1)\right]^2}{\frac{1}{4}(\rho+1)^2} = 1$$

implying the existence of an attractor since $\sqrt{x^2 + y^2 + \overline{z}^2}$ is the distance of the point $(x, y, \overline{z})$ from the origin. [3 marks]

[Past exam question. All covered in tutorials, lectures and classwork]