MATH322

Instructions to candidates

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be taken into account.

1. Consider the dynamical system given by iterations of a function $f(x)$. Define a fixed point of this system and derive the general condition for its stability (you need not discuss the special cases that may arise when the general condition is inconclusive). [4 marks]

Consider the dynamical system given by iterations of a function $f(x)$ for the following three cases (where $x \in [0,1]$):

$$f(x) = x + 1/\sqrt{2} \quad \mathrm{mod}(1)$$
$$f(x) = \sqrt{2}x \quad \mathrm{mod}(1)$$
$$f(x) = x^{1/2}$$

In each case find any fixed points and determine their stability. [8 marks]

Discuss the sensitivity to the initial condition in each case by evaluating the Lyapounov exponent or otherwise. [8 marks]

2. Consider the dynamical system given by iterations of a function

$$f(x) = 1 - 0.8x^2$$

for $x \in [0,1]$.

Find any fixed points of the sytem and determine their stability. [6 marks]

Now consider $F(x) = f(f(x))$ and sketch the graphs of $f(x)$ and of $F(x)$. [4 marks]

Consider now the dynamical system defined by $F(x) = f(f(x))$. Show that the equation for a fixed point, which is not a fixed point of the system defined by $f(x)$, is given by $-0.64x^2 + 0.8x - 0.2 = 0$ and find the location of these fixed points. Discuss their stability. [8 marks]

Give a brief argument why the system defined by $f(x)$ will not show chaos. [2 marks]

3. (i) Consider a square divided into 9 smaller squares arranged as a 3×3 pattern. Consider the operation Q of removing 4 of these smaller squares, namely those at the centres of each side. Apply Q initially to a unit length square to leave 5 smaller squares. Then apply Q again to each of these smaller squares. The process is repeated indefinitely.

Discuss whether the resulting set is self-similar under scale changes and find its capacity dimension. [6 marks]

(ii) A dynamical system on [0,1] is given by

$$x_{n+1} = f(x_n) \quad \text{where}$$

$$f(x) = 0 \quad \text{for } \frac{1}{5} < x \leq \frac{3}{5}$$

$$f(x) = 5x \mod(1), \quad \text{otherwise.}$$

Sketch $f(x)$.

Show that the fixed points of this system are unstable.

Consider the set S of initial points x_0 for which $x_n \neq 0$ as $n \to \infty$. Obtain a description of S and use it to find the capacity dimension of S.

Give an example in base 5 of an initial value x_0 for which the system will show periodic behaviour. [14 marks]

4. Consider a dynamical system defined by iterates of the functional relationships

$$x_{n+1} = f(x_n, y_n) , \qquad y_{n+1} = g(x_n, y_n)$$

with

$$f(x, y) = 1 + y - x^2 , \qquad g(x, y) = x .$$

Consider the image under one iteration of the system of the x-axis, obtain an algebraic expression for this image and sketch it. [5 marks]

Now consider the system for all (x, y) and find all fixed points. Linearise the system about them and discuss their stability. [10 marks]

Define the average Lyapounov exponents for a dynamical system with two variables. Show that for the above system, the sum of the Lyapounov exponents is constant and find an expression for it. [5 marks]

5. Discuss briefly some possible bifurcations that can occur as a parameter is varied in a dynamical system described by two autonomous coupled differential equations. [2 marks]

Consider the dynamical system described by

$$\frac{dx}{dt} = -(y^2 - 1)x - y$$

$$\frac{dy}{dt} = x.$$

Find any fixed points and determine their stability. [8 marks]

Consider the trajectories as they pass through the four points $(x,\ y) = (\pm 1,\ 0),\ (0,\ \pm 1)$ and sketch the directions of the tangents to the trajectories. [6 marks]

Give a plausibility argument for the nature of the solution. Discuss how a Poincaré section might be used to investigate the nature of the solution for this case. [4 marks]

6. Describe briefly the types of dynamically invariant set that can arise from a dynamical system defined by three coupled autonomous differential equations. Show also that the trajectories cannot cross in phase space. [5 marks]

Consider the equations

$$\frac{dx}{dt} = -x + y$$

$$\frac{dy}{dt} = 2x - y - xz$$

$$\frac{dz}{dt} = xy - z$$

Show that this system of equations describes a dissipative system. [2 marks]

Show that the origin is a fixed point and find the other two fixed points of these equations. [3 marks]

Consider first the fixed point at the origin and show that it is a saddle point. [5 marks]

For each of the other two fixed points, determine their stability, given that one of the eigenvalues is -2 in each case. [5 marks]

7. Discuss briefly all of the three following topics:

(i) The Feigenbaum constants arising in the period doubling route to chaos. [7 marks]

(ii) Express the equation for the forced, damped, circular pendulum in terms of autonomous coupled non-linear first-order differential equations and give arguments why chaotic solutions may exist. [6 marks]

(iii) A model that is able to generate a power-law behaviour of the type that might be relevant to the study of earthquakes. [7 marks]