2MA62

Instructions to candidates

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be taken into account.

1. Explain what is meant by chaos with reference to a one dimensional dynamical system with discrete time.

Consider the function defined on the domain [0,1] as

$$f(x) = \mu x(1-x)$$

where μ is a real constant $0 < \mu \leq 4$, and consider a dynamical system given by iterations of the function

$$x_{n+1} = f(x_n).$$

Derive the general condition on $f(x)$ for a fixed point of a system to be stable.

For $0 < \mu \leq 4$, find the fixed points of the given system and discuss their stability.

Find the value of μ for which the fixed point is super-stable ($df/dx = 0$).

Consider now the system defined by $f(f(x))$. Show that this system has a superstable fixed point when $\mu = 1 + \sqrt{5}$.

2. (i) Consider a square divided into $N \times N$ smaller squares where N is a positive integer. Consider the operation Q of removing a fraction f of these smaller squares. Apply Q initially to a unit length square to leave $(1-f)N^2$ smaller squares. Then apply Q again to each of these smaller squares. The process is repeated indefinitely.

Describe the conditions on the removal operation above under which the resulting set is self-similar under scale changes.

Assuming that it has been constructed in a self-similar manner, find at least one pair of values of f and N such that the resulting set has capacity dimension 1.5.

(ii) A dynamical system on [0,1] is given by

$$x_{n+1} = f(x_n) \quad \text{where}$$

$$f(x) = 4x \text{ for } x < 0.25$$

$$f(x) = 4x - 2 \text{ for } 0.5 < x < 0.75$$

$$f(x) = 0 \text{ otherwise.}$$

Sketch $f(x)$.

Consider the set S of initial points x_0 for which $x_n \neq 0$ as $n \to \infty$. Obtain a description of S and use it to show that the capacity dimesion of S is 0.5.

3. Consider a dynamical system defined by iteration of the function defined on the domain [0,1] as

$$f(x) = x + \Omega - \frac{K}{2\pi}\sin(2\pi x) \mod(1),$$

where Ω and K are real constants.

Discuss the behaviour of this system when $K = 0$ and for the two cases $\Omega = 0.5$ and $\Omega = \sqrt{2} - 1$.

Consider now $K = 0.5$ and show that the system is invertible for any Ω. Give a plausibility argument whether there can be any chaotic behaviour in this case.

From now on, consider the case where $\Omega = 0.5$ and K is close to π.

Show that the system for $K = \pi$ has a fixed point at $x = 0.25$ and find any other fixed points.

When K is just below π ($K = \pi - \epsilon$ with $0 < \epsilon << 1$), approximate $f(x)$ as a quadratic series about $x = 0.25$. Sketch the map $f(x)$ for x near 0.25 and show that the dynamical system may have intermittent behaviour since the number of steps spent near $x = 0.25$ increases as $1/\sqrt{\epsilon}$ as $\epsilon \to 0$.

4. Consider a dynamical system defined by iterates of the functional relationships

$$x_{n+1} = f(x_n, y_n) \ , \qquad y_{n+1} = g(x_n, y_n)$$

with

$$f(x, y) = 1 + y - Cx^2 \ , \qquad g(x, y) = Bx \ .$$

where $B = 0.3$ and $C = 1$.

Show that this system is invertible by explicitly obtaining the inverse map.

Consider the image under one iteration of the system of the square with boundaries $x = \pm 1$, $y = \pm 1$. Obtain algebraic expressions for the boundaries of this image and sketch them. Find the area of the image.

Now consider the system for all (x, y) and find all fixed points. Linearise the system about them and discuss their stability.

Define the average Lyapounov exponents for a dynamical system with two variables. Show that for the above system, the sum of the Lyapounov exponents is constant and find an expression for it.

5. Discuss briefly some possible bifurcations that can occur as a parameter is varied in a dynamical system described by two autonomous coupled differential equations.

Consider the equations describing the dynamical behaviour of some chemical reactions:

$$\frac{dx}{dt} = 1 - (b+1)x + x^2 y$$

$$\frac{dy}{dt} = bx - x^2 y,$$

where x, y are concentrations of components in the system and b is a real positive parameter.

Find any fixed points of the system and discuss their stability. Distinguish all different regions of behaviour as b varies from 0 to infinity.

For the particular case when $b = 3$, consider trajectories which cross the line $y = 3$ (for $x > 0$) and sketch the directions of the tangents to these trajectories as they cross the line.

Give plausibility arguments that the system has a stable limit cycle when $b = 3$.

6. Consider the equations

$$\frac{dx}{dt} = -x + 2y$$

$$\frac{dy}{dt} = x - y - xz$$

$$\frac{dz}{dt} = xy - z$$

Show that this system of equations describes a dissipative system.

Show that the trajectories cannot cross in phase space.

Show that the origin is a fixed point and find the other two fixed points of these equations.

Consider first the fixed point at the origin: linearise the equations about this fixed point and show that it is a saddle point.

For each of the other two fixed points, linearise the equations around them and find the characteristic equations for the eigenvalues. Show that in each case, these eigenvalues are -2 and $(-1 \pm \sqrt{3}i)/2$.

Without evaluating the eigenvectors, describe briefly the motion of trajectories which are near each of these two fixed points.

7. Discuss models for fractal pattern formation. In particular give examples of mathematical models which may be appropriate to snowflake growth and to earthquake energy distribution.

Discuss briefly self organised criticality.