MAY 2006 EXAMINATIONS

Degree of Bachelor of Engineering : Year 2
Degree of Master of Engineering : Year 2

MATHEMATICS AND NUMERICAL METHODS FOR CIVIL ENGINEERS
(here section A only)

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to SIX questions,
of which no more than FIVE answers must be from the Section A and no less than ONE must be from the Section B.
Candidates should submit answers to Sections A and B in **separate books**.

SECTION A

1. (a) Find the adjoint, adj $(\mathbf{A})$, determinant, det $\mathbf{A}$, and inverse, $\mathbf{A}^{-1}$, of the square matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & -1 & 1 \end{bmatrix}.$$

(7 marks)

(b) Using your result from part (2a), find the solution to the system of simultaneous equations

$$\begin{aligned} -2y + z &= -3, \\ x + y - 2z &= -1, \\ -2x - y + z &= 1. \end{aligned}$$

(6 marks)

(c) Solve the same system of equations as in part (**??**), using Cramer's rule.

(4 marks)

2. (a) Use Gaussian elimination method to show that one of the following systems has no solutions and the other has an infinite number of solutions. (HINT: find the rank of $\mathbf{A}$ and $\mathbf{A}|\mathbf{b}$ and compare with n, the number of unknowns):

(i) $\begin{bmatrix} 2 & -3 & -1 \\ 2 & 1 & -3 \\ 4 & -4 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$

(7 marks)

(ii) $\begin{bmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 4 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$

(7 marks)

(b) Find the general solution of whichever of the above systems is consistent and write the solution in parametric form.

(3 marks)

3. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

- Write down its characteristic polynomial. One of its eigenvalues is $\lambda_1 = -1$. Find the other two, λ_2 and λ_3.

  (5 marks)

- Find an eigenvector for each of the three eigenvalues λ_1, λ_2, λ_3.

  (6 marks)

- Find the unit length (normalised) eigenvectors corresponding to the three eigenvalues λ_1, λ_2 and λ_3.

  (3 marks)

- Consider the following system of ordinary differential equations:

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= -x + y + 3z \\ \mathrm{d}y/\mathrm{d}t &= 2y + z \\ \mathrm{d}z/\mathrm{d}t &= y + 2z \end{aligned}$$

  Using your previous results, write down a general solution of this system, depending on three arbitrary contants C_1, C_2 and C_3.

  (3 marks)

4. (a) Find the stationary point of the function

$$f(x, y) = x^2 + 2y^2 + 3xy - 2x - 3y$$

   Classify this point.

   (9 marks)

   (b) Find an equation of the tangent plane at the point $(x, y) = (1, 1)$ for the graph of the function $f(x)$ defined above in part (**??**). Find a normal vector to that plane. Calculate also the directional derivative of this function in the direction $\theta = -45°$ at the same point $(1, 1)$.

   (8 marks)

5. (a) Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy} and f_{yy} of the function

$$f(x,y) = e^{2x} \sin y.$$

(4 marks)

(b) Using your result from part (5a), find the Taylor series at $(0, \pi/2)$ for f up to and including terms quadratic in the increments δx and δy.

(9 marks)

(c) Use the approximation to the Taylor series found in part (5b) to obtain linear and quadratic approximation for $f(0.1, 1.5)$, with 3 decimal places.

(4 marks)

6. (a) The function $g(x)$ is periodic, with period $p = 2L = 2$, and has the Fourier series expansion

$$g(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\pi x) + b_n \sin(n\pi x)).$$

State the formulae for the Fourier coefficients, a_0, a_n, $n = 1, 2, \ldots$ and b_n, $n = 1, 2, \ldots$, valid for this period.

(5 marks)

(b) The function $g(x)$ is defined by

$$g(x) = \begin{cases} -2x, & -1 \le x \le 0, \\ 2x, & 0 \le x \le 1, \\ g(x \pm 2), & \text{for all } x. \end{cases}$$

Sketch the graph of $g(x)$ for $-3 < x < 3$. Give the definition of an even function. Explain what special features a Fourier series of an even function has. Explain why the function $g(x)$ defined above is even.

(4 marks)

(c) Find the Fourier series of the function $g(x)$ defined above. You may use the following result: $\int x \cos(kx)\,\mathrm{d}x = \frac{x}{k}\sin(kx) + \frac{1}{k^2}\cos(kx)$, where $k \neq 0$ is a constant. Write out this series explicitly up to terms with $\cos(5\pi x)$ and $\sin(5\pi x)$.

(8 marks)

SECTION B