MATH295. January 2006. Solutions

1. (a) $dc_L = \dfrac{\partial c_L}{\partial \lambda} d\lambda + \dfrac{\partial c_L}{\partial \mu} d\mu + \dfrac{\partial c_L}{\partial \rho} d\rho$ [1]

$\dfrac{\partial c_L}{\partial \lambda} = \dfrac{1}{2\rho}\left(\dfrac{\lambda + 2\mu}{\rho}\right)^{-1/2}$ [1]

$\dfrac{\partial c_L}{\partial \mu} = \dfrac{1}{\rho}\left(\dfrac{\lambda + 2\mu}{\rho}\right)^{-1/2}$ [1]

$\dfrac{\partial c_L}{\partial \rho} = -\dfrac{(\lambda + 2\mu)}{2\rho}\left(\dfrac{\lambda + 2\mu}{\rho}\right)^{-1/2}$ [1]

$\dfrac{dc_L}{c_L} = dc_L\left(\dfrac{\lambda + 2\mu}{\rho}\right)^{-1/2} = \dfrac{1}{2}\dfrac{1}{\lambda + 2\mu} + \dfrac{d\mu}{\lambda + 2\mu} - \dfrac{1}{2}\dfrac{d\rho}{\rho}$ [2]

$|d\lambda| \leq 0.02$, $|d\mu| \leq 0.03$, $|d\rho| \leq 0.01$,

$\left.\dfrac{\partial c_L}{\partial \lambda}\right|_{(\lambda=1,\mu=\frac{1}{2},\rho=2)} = \dfrac{1}{4}$; $\left.\dfrac{\partial c_L}{\partial \mu}\right|_{(\lambda=1,\mu=\frac{1}{2},\rho=2)} = \dfrac{1}{2}$; $\left.\dfrac{\partial c_L}{\partial \rho}\right|_{(\lambda=1,\mu=\frac{1}{2},\rho=2)} = -\dfrac{1}{4}$ [2]

When $\lambda = 1$, $\mu = \frac{1}{2}$, $\rho = 2 \Rightarrow \dfrac{dc_L}{c_L} = \dfrac{1}{4}\dfrac{2}{100} + \dfrac{1}{2}\dfrac{3}{100} - \dfrac{1}{4}\dfrac{1}{100} = \dfrac{7}{400}$

$$\% \text{ error } = \frac{dc_L}{c_L} \times 100 = \frac{7}{4}\% = 1.75\%. \qquad [2]$$

[10 marks]

b) $f(x,y) = x^3 + y^3 + 3x^2 - 3y^2$

Stationary points, given by $f_x(x,y) = f_y(x,y) = 0$.

$f_x(x,y) = 3x^2 + 6x = 0 \iff 3x(x+2) = 0$, so $x = 0$ or $x = -2$

$f_y(x,y) = 3y^2 - 6y = 0 \iff 3y(y-2) = 0$, so $y = 0$ or $y = 2$

4 possible stationary points are $(0,0)$, $(0,2)$, $(-2,0)$ and $(-2,2)$ [4]

Discriminant $D(a,b) = f_{xy}^2(a,b) - f_{xx}(a,b)f_{yy}(a,b)$

$f_{xy} = 0$;

$f_{xx}(x,y) = 6x + 6$;

$f_{yy}(x,y) = 6y - 6$. [2]

At $(0,0)$ $f_{xx}(0,0) = 6$, $f_{yy}(0,0) = -6 \Rightarrow D(0,0) = 36 > 0$, so $(0,0)$ is a saddle point. [1]

At $(0,2)$ $f_{xx}(0,2) = 6$, $f_{yy}(0,2) = 6 \Rightarrow D(0,2) = -36 < 0$, and $f_{xx}(0,2) > 0$ so $(0,2)$ is a minimum. [1]

At $(-2,0)$ $f_{xx}(-2,0) = -6$, $f_{yy}(-2,0) = -6 \Rightarrow D(-2,0) = -36 < 0$, and $f_{xx}(0,2) < 0$ so $(-2,0)$ is a maximum. [1]

At $(-2,2)$ $f_{xx}(-2,2) = -6$, $f_{yy}(-2,2) = 6 \Rightarrow D(-2,2) = 36 > 0$, so $(-2,2)$ is a saddle point. [1]

[10 marks]

2. $F(x,y,z) = z^2 + zx^2y + 6xy - \log(x^2+y^2) = 0$

(a) $z_x = -\dfrac{F_x}{F_z} = -\dfrac{2xzy + 6y - \frac{2x}{x^2+y^2}}{2z + x^2y} = \dfrac{2x - (x^2+y^2)(2xyz + 6y)}{(x^2+y^2)(2z+x^2y)}$ [3]

$z_y = -\dfrac{F_y}{F_z} = -\dfrac{x^2z + 6x - \frac{2y}{x^2+y^2}}{2z + x^2y} = \dfrac{2y - (x^2+y^2)(x^2z + 6x)}{(x^2+y^2)(2z+x^2y)}$ [3]

[6 marks]

(b) $z_x(0,e,\sqrt{2}) = \dfrac{-e^2 \cdot 6e}{2\sqrt{2}e^2} = -\dfrac{3\sqrt{2}}{2}e$ [1]

$z_y(0,e,\sqrt{2}) = \dfrac{2e - 0}{2\sqrt{2}e^2} = \dfrac{\sqrt{2}}{2e}$ [1]

Tangent plane: $z - \sqrt{2} = -\dfrac{3\sqrt{2}}{2}e(x-0) + \dfrac{\sqrt{2}}{2e}(y-e) \iff$

$$z + \frac{3\sqrt{2}}{2}ex - \frac{\sqrt{2}}{2e}y + \frac{\sqrt{2}}{2} - \sqrt{2} = 0 \qquad [2]$$

[4 marks]

(c) $F_z(x,y,z) = 2z + x^2y$. If $2z + x^2y = 0$ then z is not defined implicitly as a function of x and y [4]

[4 marks]

(d) $x = \cos(t)$, $y = \sin(t)$, $z = t$

$\dfrac{dF}{dt} = \dfrac{\partial F}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial F}{\partial z}\dfrac{dz}{dt}$

$\dfrac{\partial F}{\partial x} = 2xzy + 6y - \dfrac{2x}{x^2+y^2}, \qquad \dfrac{dx}{dt} = -\sin(t)$ [1]

$\dfrac{\partial F}{\partial y} = x^2z + 6x - \dfrac{2y}{x^2+y^2}, \qquad \dfrac{dy}{dt} = \cos(t)$ [1]

$\dfrac{\partial F}{\partial z} = 2z + x^2y, \qquad \dfrac{dz}{dt} = 1$ [1]

$\dfrac{dF}{dt} = (2t\sin(t)\cos(t) + 6\sin(t) - 2\cos(t))(-\sin(t)) +$
$+(t\cos^2(t) + 6\cos(t) - 2\sin(t))\cos(t) + 2t + \cos^2(t)\sin(t)$
$= -2t\sin^2(t)\cos(t) - 6\sin^2(t) + t\cos^3(t) + 6\cos^2(t) + 2t + \cos^2(t)\sin(t)$
$= 2t - 2\sin^2(t)(t\cos(t) + 3) + \cos^2(t)(\sin(t) + t\cos(t) + 6)$ [3] [6 marks]

3. (a) $f(x,y) = y\cos(x^2y)$ near $(1,2\pi)$

$$\begin{aligned} p_2(x,y) &= f(1,2\pi) + (x-1)f_x(1,2\pi) + (y-2\pi)f_y(1,2\pi) \\ &+ \tfrac{1}{2}\left[(x-1)^2 f_{xx}(1,2\pi) + 2(x-1)(y-2\pi)f_{xy}(1,2\pi) + (y-2\pi)^2 f_{yy}(1,2\pi)\right] \end{aligned} \quad [2]$$

Construct the table

$f(1,2\pi) = 2\pi$

$f_x(x,y) = -2xy^2\sin(x^2y), \Rightarrow f_x(1,2\pi) = 0$ [2]

$f_y(x,y) = \cos(x^2y) - yx^2\sin(x^2y), \Rightarrow f_y(1,2\pi) = 1$ [2]

$f_{xx}(x,y) = -2y^2\sin(x^2y) - 4x^2y^3\cos(x^2y), \Rightarrow f_{xx}(1,2\pi) = -4\cdot 8\pi^3 = -32\pi^3$ [2]

$f_{xy}(x,y) = -4xy\sin(x^2y) - 2x^3y^2\cos(x^2y), \Rightarrow f_{xy}(1,2\pi) = -2\cdot 4\pi^2 = -8\pi^2$ [2]

$f_{yy}(x,y) = -x^2\sin(x^2y) - x^2\sin(x^2y) - x^4y\cos(x^2y), \Rightarrow f_{yy}(1,2\pi) = -2\pi$ [2]

$$p_2(x,y) = 2\pi + (y-2\pi) + \frac{1}{2}\left[-32\pi^3(x-1)^2 - (x-1)(y-2\pi)16\pi^2 - 2\pi(y-2\pi)^2\right] \quad [2]$$

[14 marks]

(b) $F(x,y,z) = x^2\cos(y+2z) + y^2ze^{-x}$, $P_0 = (2,0,\pi)$, $\mathbf{t} = (1,-1,3)$

$\nabla F(x,y,z) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} =$

$= (2x\cos(y+2z) - y^2ze^{-x},\ 2yze^{-x} - x^2\sin(y+2z),\ y^2e^{-x} - 2x^2\sin(y+2z))$ [2]

$\nabla F(2,0,\pi) = (4,0,0)$

$\mathbf{t} = (1,-1,3)$ $|\mathbf{t}| = \sqrt{11} \Rightarrow \mathbf{t}_u = \left(\frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{3}{\sqrt{11}}\right)$ [2]

$D_{\mathbf{t}_u}F(2,0,\pi) = \nabla F\cdot\mathbf{t}_u = \dfrac{4}{\sqrt{11}}$ [2]

[6 marks]

4. $f(t) = |x|, -4\pi < t \le 4\pi$ and is periodic of period $T = 2\pi$.

PSfrag replacements

-4π

-3π

-2π

$-\pi$

π

2π

3π

4π

2π

$f(t)$

t

[4 marks]

(b) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$ [2]

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$ [2]

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$ [2]

[6 marks]

(c) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|dt = \frac{2}{\pi}\int_{0}^{\pi} tdt$, since $|t| = \begin{cases} t, & t \geq 0, \\ -t, & < 0 \end{cases}$ is an even function,

so $a_0 = \frac{2}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi} = \pi$ [2]

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\cos(nt)dt,$

and since $|t|\cos(nt)$ is an even function, $a_n = \frac{2}{\pi}\int_0^{\pi} t\cos(nt)dt = \frac{2}{\pi}\left(\left[\frac{t}{n}\sin(nt)\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nt)dt\right)$,

from integration by parts ($u = t$, $dv = \cos(nt)$)

Since $\sin(n\pi) = \sin(0) = 0$, $a_n = \frac{2}{\pi n}\int_0^{\pi}(-\sin(nt))dt = \frac{2}{\pi n^2}\left[\cos(nt)\right]_0^{\pi} = \frac{2}{\pi n^2}\left[\cos(n\pi) - 1\right]$ [3]

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\sin(nt)dt = 0$, since $|t|\sin(nt)$ is an odd function [1]

Thus, the Fourier series for $f(t)$ is given by $f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[\cos(n\pi) - 1]\cos(nt)$ [2];

$\cos(n\pi) = (-1)^n$

f n is even, i.e. $n = 2k$, $k = 1, 2, \ldots$, $[\cos(n\pi) - 1] = [(-1)^{2k} - 1] = 0$.

If n is odd, i.e $n = 2k - 1$, $k = 1, 2, \ldots$. $[(-1)^{2k-1} - 1] = -2$, and thus

$$f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-4}{\pi(2n-1)^2}\cos((2n-1)t) \qquad [2]$$

[10 marks]

5. $\overline{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt, \quad s > 0$

(a) $\mathcal{L}\left\{\frac{dx}{dt}(t)\right\} = \int_0^\infty f(t)e^{-st}dt = \left[x(t)e^{-st}\right]_0^\infty + s\int_0^\infty x(t)e^{-st}dt$, using integration by parts ($u = e^{-st}, dv = \frac{dx}{dt}dt$) [3]

Since $x(t)$ is exponentially bounded, $|x(t)| \leq Me^{bt} \Rightarrow |x(t)e^{-st}| \leq Me^{(b-s)}t$, thus $\lim_{t\to\infty} x(t)e^{-st} = 0, \quad (s > b > 0)$ [2]

$\lim_{t\to 0^+} x(t)e^{-st} = x(0)$, since $x(t)$ is continuous for $t \geq 0$ [1].

Thus, $[x(t)e^{-st}]_0^\infty = -x(0)$, $s > 0$ [1]

$s\int_0^\infty x(t)e^{-st}dt = s\overline{x}(s) = s\mathcal{L}\{x(t)\}$ and

$$\mathcal{L}\{\frac{dx}{dt}(t)\} = s\mathcal{L}\{x(t)\} - x(0) \qquad [2]$$

Hence, $\mathcal{L}\{\frac{d^2x}{dt^2}(t)\} = s\mathcal{L}\{\frac{dx}{dt}(t)\} - \frac{dx}{dt}(0) = s^2\mathcal{L}\{x(t)\} - sx(0) - \frac{dx}{dt}(0)$ [2]

[10 marks]

(b) $\frac{d^2x}{dt^2} - 3\frac{dx}{dt}(t) - 4x = e^{-2t}, \quad x(0) = 1, \quad \frac{dx}{dt}(0) = 1$

Take the LT, $\begin{cases} \mathcal{L}\{\frac{d^2x}{dt^2}(t)\} = s\mathcal{L}\{\frac{dx}{dt}(t)\} - \frac{dx}{dt}(0) = s^2\mathcal{L}\{x(t)\} - sx(0) - \frac{dx}{dt}(0) \\ \mathcal{L}\{\frac{dx}{dt}(t)\} = s\mathcal{L}\{x(t)\} - x(0) \\ \mathcal{L}\{e^{-2t}\} = \frac{1}{s+2} \end{cases}$ [1]

Thus, $s^2\overline{x}(s) - sx(0) - \frac{dx}{dt}(0) - 3(s\overline{x}(s) - x(0)) - 4\overline{x}(s) = \frac{1}{s+2}$ [2]

$\overline{x}(s)(s^2 - 3s - 4) + 4 - s = \frac{1}{s+2} \Rightarrow \overline{x}(s)(s^2 - 3s - 4) = \frac{1}{s+2} + s - 4 = \frac{s^2 - 2s - 7}{s+2}$

and

$$\overline{x}(s) = \frac{s^2 - 2s - 7}{(s+2)(s^2 - 3s - 4)} \qquad [2]$$

$s^2 - 3s - 4 = (s-4)(s+1) \Rightarrow \overline{x}(s) = \frac{s^2 - 2s - 7}{(s+2)(s^2 - 3s - 4)} = \frac{A}{s-4} + \frac{B}{s+1} + \frac{C}{s+2}$ [1]

$s^2 - 2s - 7 = A(s+1)(s+2) + B(s-4)(s+2) + C(s-4)(s+1)$

$s = 4 \Rightarrow 1 = 30A; \quad A = \frac{1}{30}$

$s = -2 \Rightarrow 1 = 6C; \quad C = \frac{1}{6}$ [2]

$s = -1 \Rightarrow -4 = -5B; \quad B = \frac{4}{5}$

Therefore $\overline{x}(s) = \frac{1}{30}\frac{1}{s-4} + \frac{4}{5}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s+2}$

From $\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha}$,

$$x(t) = \frac{1}{30}e^{4t} + \frac{4}{5}e^{-t} + \frac{1}{6}e^{-2t} \qquad [2]$$

[10 marks]