THE UNIVERSITY
of LIVERPOOL

SECTION A

1. (i) Solve the following system of linear equations

$$\begin{aligned} 3x + 3y + 5z &= 25 \\ x + 2y + 3z &= 12 \\ x + 3y - 2z &= -16 \end{aligned}$$

[15 marks]

(ii) Determine whether the following matrix is invertible. (You are not required to compute the inverse matrix explicitly).

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & 5 \end{pmatrix}$$

[5 marks]

(iii) Let A, B, C be square matrices, such that

$$ABC = CB$$

Assume that the matrices B and C are invertible. Solve the above matrix equation for A and compute $\det(A)$. [5 marks]

2. (i) Find the eigenvalues and eigenvectors of the following matrix:

$$\begin{pmatrix} 8 & -2 & 0 \\ -1 & 7 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

[10 marks]

(ii) Use your results to find the solution of the differential equations

$$\begin{aligned} \frac{dx}{dt} &= 8x - 2y \\ \frac{dy}{dt} &= -x + 7y \\ \frac{dz}{dt} &= -3z \end{aligned}$$

given that $x(0) = 0$, $y(0) = -3$, $z(0) = 1$. [9 marks]

(iii) Compute the second derivatives of the function

$$f(x, y) = (5x^3y - 4x^2y^2)e^{3xy}$$

[6 marks]

3.

The variable y is the solution of the differential equation

$$\frac{dy}{dx} = \frac{5y}{8x}$$

where $y = 1$ for $x = 1$.

(i) Find the quadratic Taylor series for y around the point $x_0 = 1$. Evaluate the series at $x = 2$ to find an approximate value, up to four decimals, for $y(2)$. [8 marks]

(ii) Use Euler's method with step length 0.2 to find the value, up to 4 decimals, of y at $x = 2$. Explain carefully what you are doing and make a list of results at all intermediate points. [10 marks]

(iii) Show that

$$y(x) = x^{\frac{5}{8}}$$

satisfies the above differential equation and initial condition. Use this to compute the value of y at $x = 2$ up to 4 decimals. [3 marks]

(iv) Compare your results and comment on them. In particular, what is the error expected when using Euler's method, and how does this compare to your results? [4 marks]

SECTION B