SECTION A

1. (i) Solve the following system of linear equations

$$\begin{aligned} 5x + 3y + 7z &= 18 \\ 3x - 2y - z &= -2 \\ x + y + z &= 4 \end{aligned}$$

[13 marks]

(ii) Determine whether the following matrix is invertible. (You are not required to compute the inverse matrix explicitly).

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

[7 marks]

(iii) Let A, B, P be square matrices, such that the following expression is well defined:

$$AP = P^T B^T$$

Assume further that the matrix P is invertible. Show that

$$B = P^T A^T P^{-1}$$

[3 marks]

Furthermore, show that

$$\det(A) = \det(B)$$

[2 marks]

2. (i) Find the eigenvalues and eigenvectors of the following matrix:

$$\begin{pmatrix} 1 & 3 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

[10 marks]

(ii) Use your results to find the solution of the differential equations

$$\begin{aligned} \frac{dx}{dt} &= x + 3y \\ \frac{dy}{dt} &= 4x + 2y \\ \frac{dz}{dt} &= -z \end{aligned}$$

given that $x(0) = 4$, $y(0) = 3$, $z(0) = 2$. [9 marks]

(iii) Find the first derivatives of the function

$$f(x, y) = (6x^2y - 3xy^2)\cos(2xy) + x^2e^{xy}\sin(2xy)$$

[6 marks]

3. The variable y is the solution of the differential equation

$$\frac{dy}{dx} = 4 + x - y$$

where $y = 2$ for $x = 0$.

(i) Find the quadratic Taylor series for y about the point $(x, y) = (0, 2)$. Evaluate the series at $x = 1$ to obtain an approximate value, up to 4 decimals, for $y(1)$. [8 marks]

(ii) Use Euler's method with step length 0.2 to find the value, up to 4 decimals, of y at $x = 1$. Explain carefully what you are doing and make a list of results at all intermediate points. [10 marks]

(iii) Show that

$$y(x) = x + 3 - e^{-x}$$

satisfies the above differential equation and the initial condition. Use this to compute the value of y at $x = 1$ up to 4 decimals. [4 marks]

(iv) Compare your results and comment on them. In particular, what is the error expected when using Euler's method, and how does this compare to your results? [3 marks]

SECTION B