Math294 September 2000 exam: solutions

All questions are problems very similar to those considered in class or in the homeworks, except for question 5a which is bookwork.

1. (a) **Question** *Evaluate the determinant*

$$\begin{vmatrix} 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ 2 & -2 & 3 & 2 \end{vmatrix}$$

Answer $\underline{-30}$

10 marks for this part

(b) **Question** *Use Cramer's rule to solve the system*

$$\begin{aligned} x + 2y + 3z &= -3 \\ x + 2y - \beta z &= 4 + 3\beta \\ x - y + 2z &= -4 \end{aligned}$$

when $\beta = -1$.

Answer If $\beta = 1$,

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = -6 \qquad \Delta_x = \begin{vmatrix} -3 & 2 & 3 \\ 1 & 2 & 1 \\ -4 & -1 & 2 \end{vmatrix} = -6$$

$$\Delta_y = \begin{vmatrix} 1 & -3 & 3 \\ 1 & 1 & 1 \\ 1 & -4 & 2 \end{vmatrix} = -6 \qquad \Delta_z = \begin{vmatrix} 1 & 2 & -3 \\ 1 & 2 & 1 \\ 1 & -1 & -4 \end{vmatrix} = 12$$

$$x = \Delta_x/\Delta = \underline{1}, \quad y = \Delta_y/\Delta = \underline{1}, \quad z = \Delta_z/\Delta = \underline{-2}.$$

11 marks for this part

(c) **Question** *At what* β *does this system have no solution?*

Answer If β is not fixed,

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & -\beta \\ 1 & -1 & 2 \end{vmatrix} = -9 - 3\beta$$

which is zero when $\beta = \underline{-3}$. With $\beta = -3$, the system becomes

$$\begin{aligned} x + 2y + 3z &= -3 \\ x + 2y + 3z &= -5 \\ x - y + 2z &= -4 \end{aligned} \tag{1}$$

and the second equation contradicts to the first, so there are no solutions

4 marks for this part

Total for this question: 25 marks

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2. (a) **Question** *$\mathbf{A}$ and $\mathbf{C}$ are two square matrices and $\det(\mathbf{A}) \neq 0$. Show that a matrix $\mathbf{B}$ satisfying*

$$\mathbf{AB} = \mathbf{CA}$$

is given by

$$\mathbf{B} = \mathbf{A}^{-1}\mathbf{CA}.$$

Answer Multiplying both parts by $\mathbf{A}^{-1}$ from the left gives

$$\mathbf{AA}^{-1}\mathbf{B} = \mathbf{A}^{-1}\mathbf{CA}$$

and the left-hand side equals

$$\mathbf{AA}^{-1}\mathbf{B} = (\mathbf{AA}^{-1})\mathbf{B} = \mathbf{IB} = \mathbf{B}$$

as required . *(verification by direct substitution is also acceptable with full credit).*

2 marks for this part

(b) **Question** *Further, show that a matrix $\mathbf{D}$ satisfying*

$$\mathbf{AD} = \mathbf{C}^2\mathbf{A}$$

is given by

$$\mathbf{D} = \mathbf{B}^2.$$

Answer The easiest way is to substitute $\mathbf{D} = \mathbf{B}^2$ into $\mathbf{AD} = \mathbf{C}^2\mathbf{A}$ which gives

$$\mathbf{AD} = \mathbf{AB}^2 = \mathbf{AA}^{-1}\mathbf{CAA}^{-1}\mathbf{CA} = (\mathbf{AA}^{-1})\mathbf{C}(\mathbf{AA}^{-1})\mathbf{CA}$$

$$= \mathbf{ICICA} = \mathbf{C}^2\mathbf{A}$$

as required

5 marks for this part

(c) **Question** *Find* $\mathbf{B} = \mathbf{A}^{-1}\mathbf{C}\mathbf{A}$, *if*

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 1 & -1 \\ 2 & -1 & 2 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ -1 & -2 & 0 \end{bmatrix} \qquad \mathbf{A}^{-1}\mathbf{C}\mathbf{A}^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ -2 & 3 & -2 \\ -2 & 2 & 1 \end{bmatrix}$$

14 marks for this part

(d) **Question** *Verify the above result by showing that* $\mathbf{AB} = \mathbf{CA}$

Answer

$$\mathbf{CA} = \begin{bmatrix} 6 & -6 & 3 \\ -2 & 2 & -2 \\ 2 & -1 & 2 \end{bmatrix} = \mathbf{AB}$$

4 marks for this part

Total for this question: 25 marks

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3. (a) **Question** *Show that* $\lambda = 1$ *is one eigenvalue of the matrix*

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 1 \\ -2 & 2 & 2 \\ 2 & -1 & -1 \end{bmatrix}$$

and find the remaining eigenvalues.

Answer Characteristic equation:

$$P(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & -1 & 1 \\ -2 & 2-\lambda & 2 \\ 2 & -1 & -1-\lambda \end{vmatrix}$$

$$= -\lambda^3 + \lambda^2 + 4\lambda - 4 = 0$$

Substitution $\lambda = 1$ gives

$$P(1) = -1 + 1 + 4 - 4 = 0$$

so $\lambda = 1$ is an eigenvalue as required. To find the other two eigenvalues, use method of undetermined coefficients to factorise $P(\lambda)$:

$$(\lambda - 1)(A\lambda^2 + B\lambda + C) = A\lambda^3 - A\lambda^2 + B\lambda^2 - B\lambda + C\lambda - C = \lambda^3 - \lambda^2 - 4\lambda + 4$$

$$\begin{aligned} A &= 1 & A &= 1 \\ -A + B &= -1 & B &= 0 \\ -B + C &= -4 & C &= -4 \\ -C &= 4 & &\text{— satisfied} \end{aligned}$$

Thus the other two eigenvalues are roots of

$$\lambda^2 - 4 = 0 \quad \Rightarrow \quad \lambda_{2,3} = \{2, -2\}$$

8 marks for this part

(b) **Question** *Find corresponding eigenvectors.*
Answer

- $\lambda_1 = 1$, $(\mathbf{A} - \lambda_1 \mathbf{I})\,\mathbf{v}_1 = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 1 & 2 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ which corresponds to

$$\begin{aligned} -x - y + z &= 0 \\ -2x + y + 2z &= 0 \\ 2x - y - 2z &= 0 \end{aligned}$$

Assuming $z = 1$, from the first two equations

$$\begin{aligned} -x - y &= -1 \\ -2x + y &= -2 \end{aligned}$$

find $x = 1$, $y = 0$; substitution into the third equation confirms this is a solution; thus $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ or any its multiple.

- $\lambda_2 = 2$, $(\mathbf{A} - \lambda_2 \mathbf{I})\,\mathbf{v}_2 = \begin{bmatrix} -2 & -1 & 1 \\ -2 & 0 & 2 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ corresponding to

$$\begin{aligned} -2y - y + z &= 0 \\ -2x + 2z &= 0 \\ 2x - y - 3z &= 0 \end{aligned}$$

a solution to which is $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$.

- $\lambda_3 = -2$, $(\mathbf{A} - \lambda_3 \mathbf{I})\,\mathbf{v}_3 = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 4 & 2 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ corresponding to

$$\begin{aligned} 2x - y + z &= 0 \\ -2x + 4y + 2z &= 0 \\ 2x - y + z &= 0 \end{aligned}$$

where the third equation coincides with the first. A solution to the system is $\mathbf{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

8 marks for this part

(c) **Question** *Write the system of differential equations for $x(t)$, $y(t)$ and $z(t)$, corresponding to the matrix differential equation*

$$\mathrm{d}\mathbf{u}/\mathrm{d}t = \mathbf{A}\mathbf{u} \qquad \textit{where} \qquad \mathbf{u}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Answer

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= -y + z \\ \mathrm{d}y/\mathrm{d}t &= -2x + 2y + 2z \\ \mathrm{d}z/\mathrm{d}t &= 2x - y - z \end{aligned}$$

2 marks for this part

(d) **Question** *Using the above results on the eigenvalues and eigenvectors of* $\mathbf{A}$, *find the general solution of this system.*

Answer The solution is

$$\mathbf{u} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + C_3 e^{\lambda_3 t}\mathbf{v}_3 = C_1 e^{t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + C_3 e^{-2t} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

or for the components,

$$\begin{aligned} x &= C_1 e^{t} - C_2 e^{2t} - C_3 e^{-2t} \\ y &= C_2 e^{2t} - C_3 e^{-2t} \\ z &= C_1 e^{t} - C_2 e^{2t} + C_3 e^{-2t} \end{aligned}$$

Question *Check your solution.*

Answer It is sufficient to check the solution for the three cases with only one of each C_1, C_2, C_3 being nonzero.

For $C_1 = 1$, $C_2 = C_3 = 0$: $x = e^t$, $y = 0$, $z = e^t$,

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t = e^t; &\quad -y + z = 0 + e^t = e^t \quad \checkmark \\ \mathrm{d}y/\mathrm{d}t = 0; &\quad -2x + 2y + 2z = -2e^t + 0 + 2e^t = 0 \quad \checkmark \\ \mathrm{d}z/\mathrm{d}t = e^t; &\quad 2x - y - z = 2e^t - 0 - e^t = e^t \quad \checkmark \end{aligned}$$

For $C_1 = 0$, $C_2 = 1$, $C_3 = 0$: $x = -e^{2t}$, $y = e^{2t}$, $z = -e^{2t}$,

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t = -2e^{2t}; &\quad -y + z = -e^{2t} - e^{2t} = -2e^{2t} \quad \checkmark \\ \mathrm{d}y/\mathrm{d}t = 2e^{2t}; &\quad -2x + 2y + 2z = 2e^{2t} + 2e^{2t} - 2e^{2t} = 2e^{2t} \quad \checkmark \\ \mathrm{d}z/\mathrm{d}t = -2e^{2t}; &\quad 2x - y - z = -2e^{2t} - e^{2t} + e^{2t} = -2e^{2t} \quad \checkmark \end{aligned}$$

For $C_1 = C_2 = 0$, $C_3 = 1$: $x = -e^{-2t}$, $y = -e^{-2t}$, $z = e^{-2t}$,

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t = 2e^{-2t}; &\quad -y + z = e^{-2t} + e^{-2t} = 2e^{-2t} \quad \checkmark \\ \mathrm{d}y/\mathrm{d}t = 2e^{-2t}; &\quad -2x + 2y + 2z = 2e^{-2t} - 2e^{-2t} + 2e^{-2t} = 2e^{-2t} \quad \checkmark \\ \mathrm{d}z/\mathrm{d}t = -2e^{-2t}; &\quad 2x - y - z = -2e^{-2t} + e^{-2t} - e^{-2t} = -2e^{-2t} \quad \checkmark \end{aligned}$$

7 marks for this part

Total for this question: 25 marks

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4. (a) **Question** *State the linear approximation to a function $f(x, y)$ in the neighbourhood of the point (x_0, y_0).*

Answer (Bookwork)

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

3 marks for this part

(b) **Question** *The function $f(x, y)$ is defined as*

$$f(x, y) = (x - y)e^{x+y}.$$

Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Answer

$$\frac{\partial f}{\partial x} = (1 + x - y)e^{x+y} \qquad \frac{\partial f}{\partial y} = (-1 + x - y)e^{x+y}$$

Question *Hence find the linear approximation to this function in the neighbourhood of the points*

- $x_0 = 0,\ y_0 = 0$;
- $x_0 = 1,\ y_0 = 1$.

Answer

- $f(x, y) \approx x - y$;
- $f(x, y) \approx e^2(x - 1) - e^2(y - 1) = e^2(x - y)$.

10 marks for this part

(c) **Question** *Find the second derivatives*

$$\frac{\partial^2 f}{\partial x^2} \text{ and } \frac{\partial^2 f}{\partial y^2}$$

for the function $f(x, y)$ defined above, and hence evaluate

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

stating your result in its simplest form.

Answer

$$\frac{\partial^2 f}{\partial x^2} = (2 + x - y)e^{x+y}, \quad \frac{\partial^2 f}{\partial y^2} = (-2 + x - y)e^{x+y},$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2(x - y)e^{x+y}$$

12 marks for this part

Total for this question: 25 marks

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5. (a) **Question** *Briefly show how to derive the explicit finite difference scheme:*

$$y_{n+1} = y_n + \frac{1}{2}\Delta_n + \frac{1}{2}hf(x_n + h, y_n + \Delta_n)$$

where

$$\Delta_n = hf(x_n, y_n)$$

for the numerical solution of the nonlinear differential equation

$$\mathrm{d}y/\mathrm{d}x = f(x, y)$$

with initial conditions

$$y(x_0) = y_0$$

*(It is **not** necessary to discuss the error in this approximation).*

Answer (Bookwork) The differential equation in the interval $(x_n, x_n + h)$ implies that

$$y_{n+1} - y_n = \int_{x_n}^{x_n+h} f(x,y)\,\mathrm{d}x,$$

and the integral can be estimated by trapezoidal rule as

$$\approx \frac{h}{2}\left(f(x_n, y_n) + f(x_n + h, y_{n+1})\right)$$

As this equation is implicit, it is not suitable for nonlinear equations. Let us estimate it by Euler's rule,

$$y^*_{n+1} = y_n + \Delta_n$$

where Δ_n is as defined in the question. Using y^*_{n+1} instead of y_{n+1} in the trapezoidal rule, we obtain that

$$y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_n + h, y_n + \Delta_n)\right) = y_n + \frac{1}{2}\Delta_n + \frac{h}{2} f(x_n + h, y_n + \Delta_n)$$

as required.

12 marks for this part

(b) **Question** *Use this scheme, with a step length $h = 0.05$, to calculate approximately $y(1.2)$ when*

$$f(x,y) = y^2 - \frac{1}{x^2} \quad \textit{and} \quad y(1) = 1.$$

Answer

n	x_n	y_n	$f(x_n, y_n)$	$\Delta_n = hf(x_n, y_n)$	$x_n + h$	$y_n + \Delta_n$	$f(x_n + h, y_n + \Delta_n)$	y_{n+1}
0	1.000	1.000000	0.000000	0.000000	1.050	1.000000	0.092971	1.002324
1	1.050	1.002324	0.097624	0.004881	1.100	1.007205	0.188017	1.009465
2	1.100	1.009465	0.192574	0.009629	1.150	1.019094	0.282409	1.021340
3	1.150	1.021340	0.286991	0.014350	1.200	1.035689	0.378208	1.037970
4	1.200	1.037970						

13 marks for this part

Total for this question: 25 marks