THE UNIVERSITY
of LIVERPOOL

SEPTEMBER 2000 EXAMINATIONS

Degree of Bachelor of Engineering : Year 2
Degree of Bachelor of Engineering : Year 3
Degree of Master of Engineering : Year 2

ENGINEERING MATHEMATICS II

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FOUR questions.
Only the best FOUR answers will be counted.

1. (a) Evaluate the determinant

$$\begin{vmatrix} 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & -1 \\ 1 & 2 & 0 & 1 \\ 2 & -2 & 3 & 2 \end{vmatrix}$$

(10 marks)

(b) Use Cramer's rule to solve the system

$$\begin{aligned} x + 2y + 3z &= -3 \\ x + 2y - \beta z &= 4 + 3\beta \\ x - y + 2z &= -4 \end{aligned}$$

when $\beta = -1$.

(11 marks)

(c) At what β does this system have no solution?

(4 marks)

2. (a) $\mathbf{A}$ and $\mathbf{C}$ are two square matrices and $\det(\mathbf{A}) \neq 0$. Show that a matrix $\mathbf{B}$ satisfying

$$\mathbf{AB} = \mathbf{CA}$$

is given by

$$\mathbf{B} = \mathbf{A}^{-1}\mathbf{CA}.$$

(2 marks)

(b) Further, show that a matrix $\mathbf{D}$ satisfying

$$\mathbf{AD} = \mathbf{C}^2\mathbf{A}$$

is given by

$$\mathbf{D} = \mathbf{B}^2.$$

(5 marks)

(c) Find $\mathbf{B} = \mathbf{A}^{-1}\mathbf{CA}$, if

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 1 & -1 \\ 2 & -1 & 2 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(14 marks)

(d) Verify the above result by showing that $\mathbf{AB} = \mathbf{CA}$

(4 marks)

3. (a) Show that $\lambda = 1$ is one eigenvalue of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 1 \\ -2 & 2 & 2 \\ 2 & -1 & -1 \end{bmatrix}$$

and find the remaining eigenvalues.

(8 marks)

(b) Find corresponding eigenvectors.

(8 marks)

(c) Write the system of differential equations for $x(t)$, $y(t)$ and $z(t)$, corresponding to the matrix differential equation

$$\mathrm{d}\mathbf{u}/\mathrm{d}t = \mathbf{A}\mathbf{u} \qquad \text{where} \qquad \mathbf{u}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

(2 marks)

(d) Using the above results on the eigenvalues and eigenvectors of $\mathbf{A}$, find the general solution of this system. Check your solution.

(7 marks)

4. (a) State the linear approximation to a function $f(x, y)$ in the neighbourhood of the point (x_0, y_0).

(3 marks)

(b) The function $f(x, y)$ is defined as

$$f(x, y) = (x - y)e^{x+y}.$$

Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Hence find the linear approximation to this function in the neighbourhood of the points

- $x_0 = 0$, $y_0 = 0$;
- $x_0 = 1$, $y_0 = 1$.

(10 marks)

(c) Find the second derivatives

$$\frac{\partial^2 f}{\partial x^2} \text{ and } \frac{\partial^2 f}{\partial y^2}$$

for the function $f(x, y)$ defined above, and hence evaluate

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

stating your result in its simplest form.

(12 marks)

5. (a) Briefly show how to derive the explicit finite difference scheme:

$$y_{n+1} = y_n + \frac{1}{2}\Delta_n + \frac{1}{2}hf(x_n + h, y_n + \Delta_n)$$

where

$$\Delta_n = hf(x_n, y_n)$$

for the numerical solution of the nonlinear differential equation

$$\mathrm{d}y/\mathrm{d}x = f(x, y)$$

with initial conditions

$$y(x_0) = y_0$$

(It is **not** necessary to discuss the error in this approximation).

(12 marks)

(b) Use this scheme, with a step length $h = 0.05$, to calculate approximately $y(1.2)$ when

$$f(x, y) = y^2 - \frac{1}{x^2} \quad \text{and} \quad y(1) = 1.$$

(13 marks)