Math294 Summer 2000 exam: solutions

All questions are problems very similar to those considered in class or in the homeworks, except for question 5a which is bookwork, as specified in the solution.

1. (a) **Question** *Evaluate the determinant*

$$\begin{vmatrix} 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & -1 \\ 1 & 2 & 0 & 1 \\ 2 & -2 & 1 & 2 \end{vmatrix}$$

Answer $\underline{-20}$

10 marks for this part

(b) **Question** *Use Cramer's rule to solve the system*

$$\begin{aligned} x + y + 3z &= 6 \\ 2x + 2y + \beta z &= -2 + 4\beta \\ x - y + 2z &= 2 \end{aligned}$$

when $\beta = 1$.

Answer If $\beta = 1$,

$$\Delta = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = -10 \qquad \Delta_x = \begin{vmatrix} 6 & 1 & 3 \\ 2 & 2 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 10$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -10 \qquad \Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 2 & 2 \\ 1 & -1 & 2 \end{vmatrix} = -20$$

$$x = \Delta_x/\Delta = \underline{-1}, \quad y = \Delta_y/\Delta = \underline{1}, \quad z = \Delta_z/\Delta = \underline{2}.$$

11 marks for this part

(c) **Question** *At what* β *does this system have no solutions?*

Answer If β is not fixed,

$$\Delta = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & \beta \\ 1 & -1 & 2 \end{vmatrix} = -12 + 2\beta$$

which is zero when $\beta = \underline{6}$. With $\beta = 6$, the system becomes

$$\begin{aligned} x + y + 3z &= 6 \\ 2x + 2y + 6z &= 22 \\ x - y + 2z &= 2 \end{aligned} \qquad (1)$$

and the second equation contradicts to the first, so there are no solutions

4 marks for this part

Total for this question: 25 marks

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2. (a) **Question** $\mathbf{A}$ *and* $\mathbf{B}$ *are two square matrices and* $\det(\mathbf{A}) \neq 0$. *Show that a matrix* $\mathbf{C}$ *satisfying*

$$\mathbf{CA} = \mathbf{AB}$$

is given by

$$\mathbf{C} = \mathbf{ABA}^{-1}.$$

Answer Multiplying both parts by $\mathbf{A}^{-1}$ gives

$$\mathbf{CAA}^{-1} = \mathbf{ABA}^{-1}$$

and the left-hand side equals

$$\mathbf{CAA}^{-1} = \mathbf{C}(\mathbf{AA}^{-1}) = \mathbf{CI} = \mathbf{C}$$

as required . *(verification by direct substitution is also acceptable with full credit).*

2 marks for this part

(b) **Question** *Further, show that a matrix* $\mathbf{D}$ *satisfying*

$$\mathbf{DA} = \mathbf{AB}^2$$

is given by

$$\mathbf{D} = \mathbf{C}^2.$$

Answer The easiest way is to substitute $\mathbf{D} = \mathbf{C}^2$ into $\mathbf{DA} = \mathbf{AB}^2$ which gives

$$\mathbf{C}^2\mathbf{A} = \mathbf{ABA}^{-1}\mathbf{ABA}^{-1}\mathbf{A} = \mathbf{AB}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B}(\mathbf{A}^{-1}\mathbf{A})$$

$$= \mathbf{ABIBI} = \mathbf{AB}^2$$

as required

5 marks for this part

(c) **Question** *Find* $\mathbf{C} = \mathbf{ABA}^{-1}$, *if*

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ -1 & -2 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Answer

$$\mathbf{A}^{-1} = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 1 & -1 \\ 2 & -1 & 2 \end{bmatrix} \qquad \mathbf{ABA}^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & -2 & 3 \end{bmatrix}$$

14 marks for this part

(d) **Question** *Verify the above result by showing that* $\mathbf{CA} = \mathbf{AB}$

Answer

$$\mathbf{CA} = \begin{bmatrix} 1 & 6 & 3 \\ 0 & 4 & 3 \\ -1 & -4 & 0 \end{bmatrix} = \mathbf{AB}$$

4 marks for this part

Total for this question: 25 marks

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3. (a) **Question** *Show that* $\lambda = 1$ *is one eigenvalue of the matrix*

$$\mathbf{A} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

and find the remaining eigenvalues.

Answer Characteristic equation:

$$P(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -1-\lambda & -1 & 1 \\ 2 & -1-\lambda & 2 \\ 2 & 1 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 - 2\lambda^2 + \lambda + 2 = 0$$

Substitution $\lambda = 1$ gives

$$P(1) = -1 - 2 + 1 + 2 = 0$$

so $\lambda = 1$ is an eigenvalue as required. To find the other two eigenvalues, use method of undetermined coefficients to factorise $P(\lambda)$:

$$(\lambda - 1)(A\lambda^2 + B\lambda + C) = A\lambda^3 - A\lambda^2 + B\lambda^2 - B\lambda + C\lambda - C = \lambda^3 + 2\lambda^2 - \lambda - 2$$

$$\begin{aligned} A &= 1 & A &= 1 \\ -A + B &= 2 & B &= 3 \\ -B + C &= -1 & C &= 2 \\ -C &= -2 & &\text{— satisfied} \end{aligned}$$

Thus the other two eigenvalues are roots of

$$\lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_{2,3} = \{-1, -2\}$$

8 marks for this part

(b) **Question** *Find corresponding eigenvectors.*
Answer

- $\lambda_1 = 1$, $(\mathbf{A} - \lambda_1 \mathbf{I})\,\mathbf{v}_1 = \begin{bmatrix} -2 & -1 & 1 \\ 2 & -2 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ which corresponds to

$$\begin{aligned} -2x - y + z &= 0 \\ 2x - 2y + 2z &= 0 \\ 2x + y - z &= 0 \end{aligned}$$

Assuming $z = 1$, from the first two equations

$$\begin{aligned} -2x - y &= -1 \\ 2x - 2y &= -2 \end{aligned}$$

find $x = 0$, $y = 1$; substitution into the third confirms this is a solution; thus $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ or any its multiple.

- $\lambda_2 = -1$, $(\mathbf{A} - \lambda_2 \mathbf{I})\,\mathbf{v}_2 = \begin{bmatrix} 0 & -1 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ corresponding to

$$\begin{aligned} -y + z &= 0 \\ 2x + 2z &= 0 \\ 2x + y + z &= 0 \end{aligned}$$

a solution to which is $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

- $\lambda_3 = -2$, $(\mathbf{A} - \lambda_3 \mathbf{I})\, \mathbf{v}_3 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ corresponding to

$$\begin{aligned} x - y + z &= 0 \\ 2x + y + 2z &= 0 \\ 2x + y + 2z &= 0 \end{aligned}$$

where the third equation coincides with the second. A solution to the system is $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

8 marks for this part

(c) **Question** *Write the system of differential equations for $x(t)$, $y(t)$ and $z(t)$, corresponding to the matrix differential equation*

$$\mathrm{d}\mathbf{u}/\mathrm{d}t = \mathbf{A}\mathbf{u} \qquad \textit{where} \qquad \mathbf{u}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

Answer

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= -x - y + z \\ \mathrm{d}y/\mathrm{d}t &= 2x - y + 2z \\ \mathrm{d}z/\mathrm{d}t &= 2x + y \end{aligned}$$

2 marks for this part

(d) **Question** *Using the above results on the eigenvalues and eigenvectors of* **A**, *find the general solution of this system.*

Answer The solution is

$$\mathbf{u} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + C_3 e^{\lambda_3 t}\mathbf{v}_3 = C_1 e^{t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + C_3 e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

or for the components,

$$\begin{aligned} x &= -C_2 e^{-t} - C_3 e^{-2t} \\ y &= C_1 e^{t} + C_2 e^{-t} \\ z &= C_1 e^{t} + C_2 e^{-t} + C_3 e^{-2t} \end{aligned}$$

Question *Check your solution.*

Answer It is sufficient to check the solution for the three cases with only one of each C_1, C_2, C_3 being nonzero.

For $C_1 = 1$, $C_2 = C_3 = 0$: $x = 0$, $y = e^t$, $z = e^t$,

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= 0; \quad -x - y + z = -e^t + e^t = 0 \quad \checkmark \\ \mathrm{d}y/\mathrm{d}t &= e^t; \quad 2x - y + 2z = -e^t + 2e^t = e^t \quad \checkmark \\ \mathrm{d}z/\mathrm{d}t &= e^t; \quad 2x + y = e^t \quad \checkmark \end{aligned}$$

For $C_1 = 0$, $C_2 = 1$, $C_3 = 0$: $x = -e^{-t}$, $y = e^{-t}$, $z = e^{-t}$,

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= e^{-t}; \quad -x - y + z = e^{-t} - e^{-t} + e^{-t} = e^{-t} \quad \checkmark \\ \mathrm{d}y/\mathrm{d}t &= -e^{-t}; \quad 2x - y + 2z = -2e^{-t} - e^{-t} + 2e^{-t} = -e^{-t} \quad \checkmark \\ \mathrm{d}z/\mathrm{d}t &= -e^{-t}; \quad 2x + y = -2e^{-t} + e^{-t} = -e^{-t} \quad \checkmark \end{aligned}$$

For $C_1 = C_2 = 0$, $C_3 = 1$: $x = -e^{-2t}$, $y = 0$, $z = e^{-2t}$,

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= 2e^{-2t}; \quad -x - y + z = e^{-2t} + 0 + e^{-2t} = 2e^{-2t} \quad \checkmark \\ \mathrm{d}y/\mathrm{d}t &= 0; \quad 2x - y + 2z = -2e^{-2t} - 0 + 2e^{-t} = 0 \quad \checkmark \\ \mathrm{d}z/\mathrm{d}t &= -e^{-t}; \quad 2x + y = -2e^{-t} + e^{-t} = -e^{-t} \quad \checkmark \end{aligned}$$

7 marks for this part

Total for this question: 25 marks

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4. (a) **Question** *State the linear approximation to a function $f(x, y)$ in the neighbourhood of the point (x_0, y_0).*

Answer

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

3 marks for this part

(b) **Question** *The function $f(x, y)$ is defined as*

$$f(x, y) = e^{x^2 - y^2}.$$

Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Answer

$$\frac{\partial f}{\partial x} = 2xe^{x^2 - y^2} \qquad \frac{\partial f}{\partial y} = -2ye^{x^2 - y^2}$$

Question *Hence find the linear approximation to this function in the neighbourhood of the points*

- $x_0 = 1,\ y_0 = 0$;
- $x_0 = 0,\ y_0 = 1$.

Answer

- $f(x, y) \approx e + 2e(x - 1)$;
- $f(x, y) \approx 1/e - 2(y - 1)/e$.

10 marks for this part

(c) **Question** *Find the second derivatives*

$$\frac{\partial^2 f}{\partial x^2} \text{ and } \frac{\partial^2 f}{\partial y^2}$$

for the function $f(x, y)$ defined above, and hence evaluate

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

stating your result in its simplest form.

Answer

$$\frac{\partial^2 f}{\partial x^2} = (2 + 4x^2)e^{x^2-y^2}, \quad \frac{\partial^2 f}{\partial y^2} = (-2 + 4y^2)e^{x^2-y^2},$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)e^{x^2-y^2}$$

12 marks for this part

Total for this question: 25 marks

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5. (a) **Question** *Briefly show how to derive the explicit finite difference scheme:*

$$y_{n+1} = y_n + \frac{1}{2}\Delta_n + \frac{1}{2}hf(x_n + h, y_n + \Delta_n)$$

where

$$\Delta_n = hf(x_n, y_n)$$

for the numerical solution of the nonlinear differential equation

$$\mathrm{d}y/\mathrm{d}x = f(x, y)$$

with initial conditions

$$y(x_0) = y_0$$

*(It is **not** necessary to discuss the error in this approximation).*

Answer (Bookwork) The differential equation in the interval $(x_n, x_n + h)$ implies that

$$y_{n+1} - y_n = \int_{x_n}^{x_n+h} f(x, y)\, \mathrm{d}x,$$

and the integral can be estimated by trapezoidal rule as

$$\approx \frac{h}{2} \left(f(x_n, y_n) + f(x_n + h, y_{n+1})\right)$$

As this equation is implicit, it is not suitable for nonlinear equations. Let us estimate it by Euler's rule,

$$y^*_{n+1} = y_n + \Delta_n$$

where Δ_n is as defined in the question. Using y^*_{n+1} instead of y_{n+1} in the trapezoidal rule, we obtain that

$$y_{n+1} = y_n + \frac{h}{2} \left(f(x_n, y_n) + f(x_n + h, y_n + \Delta_n)\right) = y_n + \frac{1}{2}\Delta_n + \frac{h}{2} f(x_n + h, y_n + \Delta_n)$$

as required.

12 marks for this part

(b) **Question** *Use this scheme, with a step length $h = 0.025$, to calculate approximately $y(2.1)$ when*

$$f(x, y) = y^2 - \frac{1}{x^2} \quad \textit{and} \quad y(2) = 1.$$

Answer

n	x_n	y_n	$f(x_n, y_n)$	$\Delta_n = hf(x_n, y_n)$	$x_n + h$	$y_n + \Delta_n$	$f(x_n + h, y_n + \Delta_n)$	y_{n+1}
0	2.000	1.000000	0.750000	0.018750	2.025	1.018750	0.793986	1.019300
1	2.025	1.019300	0.795107	0.019878	2.050	1.039178	0.841936	1.039763
2	2.050	1.039763	0.843153	0.021079	2.075	1.060842	0.893131	1.061466
3	2.075	1.061466	0.894457	0.022361	2.100	1.083828	0.947925	1.084496
4	2.100	1.084496						

13 marks for this part

Total for this question: 25 marks