MATH293 January 2004 exam: solutions

1. (a) **Question** *Find, without using the Laplace transform, the general solution of the differential equation:*

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0.$$

Answer The characteristic equation is

$$\lambda^2 + 4\lambda + 3 = 0,$$

and its roots are

$$\lambda_{1,2} = \{-1, -3\},$$

(positive discriminant, two real roots). Hence the general solution of homogeneous equation is

$$y = \boxed{C_1 e^{-x} + C_2 e^{-3x}}.$$

7 marks for this part

(b) **Question** *Using the method of undetermined coefficients, find a particular integral of the differential equation*

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 10\sin x.$$

Hence write down the general solution of the equation.

Answer The free term here is a trigonometric function, the trial solution to be a similar trigonometric function. Besides, a sin in the trial solution implies a cos with the same argument but different coefficient. So,

$$y_p = A\cos x + B\sin x.$$

Then

$$y_p' = -A\sin x + B\cos x,$$
$$y_p'' = -A\cos x - B\sin x$$

Substitution into the equation gives

$$\begin{aligned}\text{LHS} = y_p'' + 4y_p'' + 3y_p &= -A\cos x - B\sin x + 4(-A\sin x + B\cos x) + 3(A\cos x + B\sin x)\\ &= (-A + 4B + 3A)\cos x + (-B - 4A + 3B) = (2A + 4B)\cos x + (-4A + 2B)\sin x\\ &= \text{RHS} = 10 \times \sin x + 0 \times \cos x.\end{aligned}$$

Equating the coefficients at the same functions of x leads to

$$\begin{aligned}[\cos x]: &\quad 2A + 4B = 0,\\ [\sin x]: &\quad -4A + 2B = 10,\end{aligned}$$

The solution of this system is

$$A = -2, \quad B = 1$$

and so the particular solution to the nonhomogeneous equation is

$$y_p = -2\cos x + \sin x.$$

The general solution of the nonhomogeneous equation is the sum of the particular solution and the complementary function (GSHE), that is,

$$y = \boxed{-2\cos x + \sin x + C_1 e^{-x} + C_2 e^{-3x}}.$$

13 marks for this part

(c) **Question** *Suggest the form of the trial solution in the method of undetermined coefficients, for the equation*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 5\mathrm{e}^{-x} + 6x^2.$$

(You are not required to do the calculations here, i.e. you should leave the coefficients in the trial solution undetermined.)

Answer By the sum rule, the trial solution is sum of partial trial solutions for each of the terms in the RHS, which are an exponential and a square. The first term is part of the complementary function, so modification rule applies. The second term generates a similar term in the trial solution plus all smaller powers of x. Thus

$$y_p = \boxed{Axe^{-x} + Bx^2 + Cx + D}.$$

5 marks for this part

Total for this question: 25 marks

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2. (a) **Question** *The function $f(x)$ is periodic, with period $p = 2L = 2$, and has the Fourier series expansion*

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x) + b_n \sin(n\pi x)).$$

State the formulae for the Fourier coefficients, a_0, a_n, $n = 1, 2, \ldots$ and b_n, $n = 1, 2, \ldots$, valid for this period.

Answer

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,\mathrm{d}x,$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,\mathrm{d}x, \qquad n = 1, 2\ldots,$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,\mathrm{d}x, \qquad n = 1, 2\ldots$$

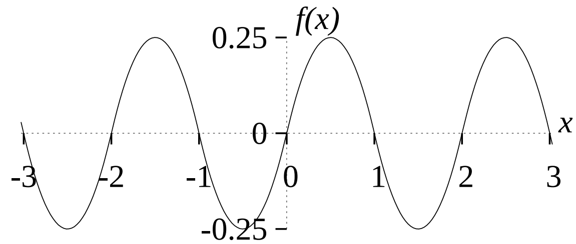
7 marks for this part

(b) **Question** *The function $f(x)$ is defined by*

$$f(x) = \begin{cases} x + x^2, & -1 \le x \le 0, \\ x - x^2, & 0 \le x \le 1, \\ f(x \pm 2), & \textit{for all } x. \end{cases}$$

Sketch the graph of $f(x)$ for $-3 < x < 3$.

Answer



Question *Give the definition of an odd function.*
Answer $f(-x) = f(x)$ for all x.
Question *Explain what special features a Fourier series of an odd function has.*
Answer It lacks the constant term and all cos terms.
Question *Explain why the function $f(x)$ defined above is odd.*
Answer Graphical: the graph is center-symmetric about the origin.
Analytical: it is odd by definition within the symmetric interval $-1 \leq x \leq 1$, and periodic with period 2 equal to the length of that interval, therefore odd everywhere.

6 marks for this part

(c) **Question** *Find the Fourier series of the function $f(x)$ defined in part (b). You may use the following result:*

$$\int (x - x^2) \sin kx = \frac{k^2(x^2 - x) - 2}{k^3} \cos kx + \frac{1 - 2x}{k^2} \sin kx$$

where k is a non-zero constant.
Answer

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\ &= \int_{-1}^{1} f(x) \sin(n\pi x) \mathrm{d}x \\ &= 2 \int_{0}^{1} f(x) \sin(n\pi x) \mathrm{d}x \\ &= 2 \int_{0}^{1} (x - x^2) \sin(n\pi x) \mathrm{d}x \end{aligned}$$

(now use the given result, for $k = n\pi$)

$$= 2 \left(\frac{(n\pi)^2(x^2 - x) - 2}{(n\pi)^3} \cos(n\pi x) + \frac{1 - 2x}{(n\pi)^2} \sin(n\pi x) \right)_0^1$$

(now use $\sin(n\pi) = 0$ for all integer n, and also that $x^2 - x = 0$ if $x = 0$ or $x = 1$)

$$= -4 \frac{1}{n^3\pi^3} (\cos(n\pi) - 1)$$

$$= \frac{4}{n^3\pi^3} (1 - (-1)^n) = \begin{cases} \frac{8}{n^3\pi^3}, & \text{for odd } n, \\ 0, & \text{for even } n, \end{cases}$$

Thus,

$$f(x) = \frac{8}{\pi^3} \sum_{n=1,3,5\ldots} \frac{1}{n^3} \sin(n\pi x)$$

Question *Write out explicitly the partial sum of this series up to and including terms with* $\cos(3\pi x)$ *and/or* $\sin(3\pi x)$.
Answer

$$f(x) \approx S_3(x) = \frac{8}{\pi^3} \left(\sin(\pi x) + \frac{1}{27} \cos(3\pi x) \right)$$

9 marks for this part

Question *Calculate the value of this partial sum at $x = 1/2$ to four decimal places and estimate the relative accuracy with which it approximates the exact value of $f(1/2)$.*
Answer

$$S_3(1/2) = \frac{8}{\pi^3}\left(\sin(\pi/2) + \frac{1}{27}\sin(3\pi/2)\right) = \frac{8}{\pi^2}\left(1 - \frac{1}{27}\right) \approx 0.2484$$

$$f(1/2) = 0.5 - 0.25 = 0.25$$

thus the relative error is $\left|\frac{f(1/2)-S_3(1/2)}{f(1/2)}\right| \approx \frac{0.0016}{0.25} \approx \boxed{6\%}$.

3 marks for this part

Total for this question: 25 marks

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3. (a) **Question** *A given function $f(t)$ can be represented using its Fourier transform $\tilde{f}(w)$ by the formula*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(w) e^{iwt}\, \mathrm{d}w.$$

Give the formula, by which the Fourier transform $\tilde{f}(w)$ can be found if the function $f(t)$ is known.
Answer

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt}\, \mathrm{d}t.$$

General structure , correct coefficient , correct limits , correct sign in the exponent .

6 marks for this part

(b) **Question** *Show that if $f(t)$ is given by*

$$f(t) = \begin{cases} t - t^2, & 0 < t < 1, \\ 0, & \textit{otherwise} , \end{cases}$$

its Fourier transform is

$$\tilde{f}(w) = \frac{(2i - w)e^{-iw} - (2i + w)}{w^3\sqrt{2\pi}}.$$

To do that, you may use the following result:

$$\int (t - t^2)e^{kt}\, \mathrm{d}t = \left(\frac{t - t^2}{k} - \frac{1 - 2t}{k^2} - \frac{2}{k^3}\right) e^{kt},$$

where k is a non-zero constant.
Answer For the given $f(t)$, the Fourier transform is

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt}\, \mathrm{d}t$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{1} (t - t^2) e^{-iwt}\, \mathrm{d}t$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{t - t^2}{-iw} - \frac{1 - 2t}{(-iw)^2} + \frac{-2}{(-iw)^3}\right) e^{-iwt}\Big|_0^1$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{1 - 1^2}{-iw} - \frac{1 - 2}{-w^2} - \frac{2}{iw^3}\right) e^{-iw} - \left(\frac{0 - 0^2}{-iw} - \frac{1 - 0}{-w^2} - \frac{2}{iw^3}\right) e^0\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(-\frac{1}{w^2} - \frac{-2i}{w^3} \right) e^{-iw} - \left(\frac{1}{w^2} - \frac{-2i}{w^3} \right) e^0 \right]$$
$$= \frac{(2i - w)e^{-iw} - (2i + w)}{w^3 \sqrt{2\pi}}.$$

as requested.

15 marks for this part

(c) **Question** *Using the results of parts (a) and (b), write down the integral which represents $f(t)$ defined in part (b).*

Answer Inverse F.t.:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(w) e^{iwt} \, \mathrm{d}w$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(2i - w)e^{-iw} - (2i + w)}{w^3} e^{iwt} \, \mathrm{d}w$$

4 marks for this part

Total for this question: 25 marks

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4. (a) **Question** *Find the function of t whose Laplace transform is:*

$$\frac{1}{(s+1)(s+3)}.$$

Answer By cover-up rule,

$$F(s) = \frac{1}{(s+1)(s+3)} = \frac{1}{(-1)+3} \cdot \frac{1}{s+1} + \frac{1}{(-3)+1} \cdot \frac{1}{s+3}$$
$$= \frac{1}{2} \frac{1}{s+1} - \frac{1}{2} \frac{1}{s+3}$$

(any other valid method equally acceptable)

$$f(t) = \mathcal{L}^{-1}[F(s)] = \boxed{\frac{1}{2}\left(e^{-t} - e^{-3t}\right)}$$

4 marks for this part

(b) **Question** *Find the function of t whose Laplace transform is:*

$$\frac{s+1}{s^2 + 4s + 5}.$$

Answer

$$F(s) = \frac{s+1}{s^2+4s+5} = \frac{s+1}{s^2+4s+4+1} = \frac{(s+2)-1}{(s+2)^2+1^2}$$
$$= \frac{(s+2)}{(s+2)^2+1^2} - \frac{1}{(s+2)^2+1^2}$$
$$f(t) = \mathcal{L}^{-1}\left[\frac{(s+2)}{(s+2)^2+1^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+1^2}\right] = e^{-2t}\cos t - e^{-2t}\sin t$$

8 marks for this part

(c) **Question** *Find, **using the Laplace Transform**, the solution of the differential equation*

$$\frac{d^2y}{dt^2} - 4y = 16e^{-2t},$$

which satisfies initial conditions

$$y(0) = 0 \quad \textit{and} \quad \frac{dy}{dt}(0) = 0.$$

Answer Let $\mathcal{L}[y] = Y$, then with account of the initial conditions,

$$\mathcal{L}[y]'' = s^2Y - sy(0) - y'(0) =$$
$$s^2Y,$$

and the subsidiary equation is

$$s^2Y - 4Y = 16\frac{1}{s+2}$$

Its solution is:

$$\begin{aligned} Y &= \frac{16}{s+2} \cdot \frac{1}{s^2-4} \\ &= \frac{16}{s+2} \cdot \frac{1}{(s-2)(s+2)} \\ &= \frac{16}{s+2}\left(\frac{1/4}{(s-2)} - \frac{1/4}{(s+2)}\right) \\ &= -\frac{4}{(s+2)^2} + \frac{4}{s+2}\frac{1}{s-2} \\ &= -\frac{4}{(s+2)^2} + \frac{1}{s-2} - \frac{1}{s+2} \end{aligned}$$

Thus

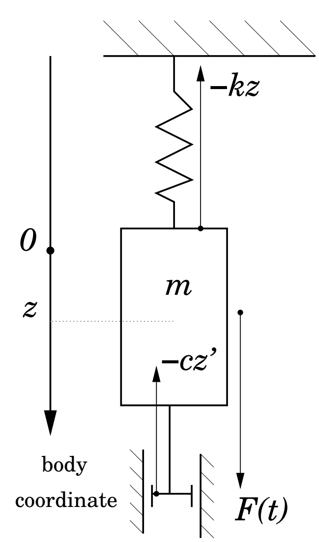
$$y = \mathcal{L}^{-1}[Y] = -4te^{-2t} - e^{-2t} + e^{2t}$$

13 marks for this part

Total for this question: 25 marks

5.

A body of mass $m = 1\,(\mathrm{kg})$ is suspended on a spring and attached to a dashpot (see the diagram). The vertical coordinate z of the body is measured with respect to its equilibrium position. Sping has the Hooke's spring constant of $k = 2\,(\mathrm{N/m})$. The frictional force acting on the body is proportional to its velocity with the coefficient $c = 3\,(\mathrm{Ns/m})$. In addition, the body is affected by a periodic external force, changing with time according to the law $F(t) = F_0\cos(\omega t)$ where $F_0 = 65\,(\mathrm{N})$ and $\omega = 2\,(\mathrm{s}^{-1})$.



(a) **Question** *Write down a differential equation for the vertical coordinate $z(t)$ of the body.*
Answer

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + 3\frac{\mathrm{d}z}{\mathrm{d}t} + 2z = 65\cos(2t)$$

5 marks for this part

(b) **Question** *Using the Laplace transform, or otherwise, find the solution to this equation for the initial conditions $z(0) = 0$, $\mathrm{d}z/\mathrm{d}t(0) = 0$.*

Answer variant 1. The characteristic equation is $\lambda^2 + 3\lambda + 2 = 0$, which has two real roots $\lambda_{1,2} = \{-1, -2\}$, and the complementary functions is

$$z_h = C_1 e^{-t} + C_2 e^{-2t} \qquad .$$

The trial solution $z_p = A\cos(2t) + B\sin(2t)$. Thus $z_p' = 2B\cos(2t) - 2A\sin(2t)$, $z_p'' = -4A\cos(2t) - 4B\sin(2t)$, substitution into the equation gives

$$(-4A + 6B + 2A)\cos(2t) + (-4B - 6A + 2B)\sin(2t) = 65\cos(2t)$$

which leads to the system

$$\begin{aligned} -2A + 6B &= 65 \\ -6A - 2B &= 0 \end{aligned}$$

the solution of which is $A = -13/4$, $B = 39/4$. Thus the general solution of the equation is

$$z = z_p + z_h = -\frac{13}{4}\cos(2t) + \frac{39}{4}\sin(2t) + C_1 e^{-t} + C_2 e^{-2t}$$

The corresponding velocity is

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{13}{2}\sin(2t) + \frac{39}{2}\cos(2t) - C_1 e^{-t} - 2C_2 e^{-2t}$$

Substituting this into the initial conditions, we get

$$\begin{aligned} z(0) = C_1 + C_2 - 13/4 &= 0 \\ z'(0) = -C_1 - 2C_2 + 39/2 &= 0 \end{aligned}$$

the solution of which is $C_1 = -13$, $C_2 = 65/4$, and the ultimate answer is

$$z(t) = -\frac{13}{4}\cos(2t) + \frac{39}{4}\sin(2t) - 13e^{-t} + \frac{65}{4}e^{-2t}$$

Answer variant 2. Let $\mathcal{L}[z(t)] = Z(s)$. Then $\mathcal{L}[z'(t)] = sZ(s) - z(0) = sZ$, $\mathcal{L}[z''(t)] = s^2 Z - sz(0) - z'(0) = s^2 Z$, $\mathcal{L}[65\cos(2t)] = \frac{65s}{s^2+4}$. Then the Laplace transform of the equation (the subsidiary equation) is

$$s^2 Z + 3sZ + 2Z = \frac{65s}{s^2 + 2^2}$$

The solution of this equation is

$$Z(s) = \frac{65s}{(s^2 + 2^2)(s^2 + 3s + 2)}$$

To find the inverse Laplace transform of $Z(s)$, we need to factorise the denominator,

$$(s^2 + 2^2)(s^2 + 3s + 2) = (s^2 + 2^2)(s + 1)(s + 2),$$

and represent $Z(s)$ as a sum of terms in the standard form,

$$Z(s) = \frac{As + B}{s^2 + 4} + \frac{C}{s + 1} + \frac{D}{s + 2}.$$

Bringing this to a common denominator produces

$$Z(s) = \frac{(A + C + D)s^3 + (B + 3A + 2C + D)s^2 + (3B + 2A + 4C + 4D)s + (2B + 8C + 4D)}{(s^2 + 4)(s + 1)(s + 2)}$$

$$= \frac{65s}{(s^2 + 4)(s + 1)(s + 2)}$$

Equating the coefficients at powers of s in the numerators, this gives the system

$$\begin{aligned} A + C + D &= 0 \\ 3A + B + 2C + D &= 0 \\ 2A + 3B + 4C + 4D &= 65 \\ 2B + 8C + 4D &= 0 \end{aligned}$$

the solution of which is

$$\begin{aligned} A &= -13/4 \\ B &= 39/2 \\ C &= -13 \\ D &= 65/4 \end{aligned}$$

Thus

$$Z(s) = -\frac{13}{4}\frac{s}{s^2 + 2^2} + \frac{39}{4}\frac{2}{s^2 + 2^2} - 13\frac{1}{s + 1} + \frac{65}{4}\frac{1}{s + 2}$$

and

$$z(t) = \mathcal{L}^{-1}[Z] = -\frac{13}{4}\cos(2t) + \frac{39}{4}\sin(2t) - 13e^{-t} + \frac{65}{4}e^{-2t}$$

15 marks for this part

(c) **Question** *Represent this solution as a sum of free movement and forced oscillations. Based on the form of free movement, or otherwise, classify this system as underdamped, critically damped or overdamped.*

Answer The forced oscillations $z_p = -\frac{13}{4}\cos(2t) + \frac{39}{4}\sin(2t)$. The free movement is $z_h = -13e^{-t} + \frac{65}{4}e^{-2t}$. This is an overdamped system.

Question *Find the coordinate z of the body at the time $t = 5\pi$ (s). Give your answer to four decimal places.*

Answer $z(5\pi) \approx -13/4 = -3.2500$. NB: if you do it using a calculator, you should take the value of π with a rather high precision. On the other hand, this answer can be obtained without calculator, if only you care to roughly estimate the value of $e^{-5\pi}$ and see that is far too small to influence the answer in the 4 d.p. requested.

5 marks for this part

Total for this question: 25 marks