THE UNIVERSITY of LIVERPOOL

JANUARY 2003 EXAMINATIONS

Bachelor of Engineering : Year 1
Bachelor of Engineering : Year 2
Bachelor of Science : Year 2
Master of Engineering : Year 2

ENGINEERING MATHEMATICS I

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FOUR questions.
Only the best FOUR answers will be counted.

1. (a) Find, without using the Laplace transform, the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0.$$

(7 marks)

(b) Using the method of undetermined coefficients, find a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 5x^2 + 3x + 3.$$

Hence write down the general solution of the equation.

(13 marks)

(c) Suggest the form of the trial solution in the method of undetermined coefficients, for the equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = (5x^2 + 3x + 3)\sin(2x).$$

(You don't have to do the calculations here, i.e. you should leave the coefficients in the trial solution undetermined.)

(5 marks)

2. (a) The function $f(x)$ is periodic, with period $p = 2L = 2$, and has the Fourier series expansion

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\pi x) + b_n \sin(n\pi x)).$$

State the formulae for the Fourier coefficients, a_0, a_n, $n = 1, 2, \ldots$ and b_n, $n = 1, 2, \ldots$, valid for this period.

(7 marks)

(b) The function $f(x)$ is defined by

$$f(x) = \begin{cases} -2x, & -1 \leq x \leq 0, \\ 2x, & 0 \leq x \leq 1, \\ f(x \pm 2), & \text{for all } x. \end{cases}$$

Sketch the graph of $f(x)$ for $-3 < x < 3$. Give the definition of an even function. Explain what special features a Fourier series of an even function has. Explain why the function $f(x)$ defined above is even.

(6 marks)

(c) Find the Fourier series of the function $f(x)$ defined above. You may use the following result: $\int x \cos(kx)\, \mathrm{d}x = \frac{x}{k} \sin(kx) + \frac{1}{k^2} \cos(kx)$, where $k \neq 0$ is any constant. Write out explicitly the partial sum of this series up to terms with $\cos(5\pi x)$ and/or $\sin(5\pi x)$.

(9 marks)

Calculate the value of this partial sum at $x = 1$ at 3 d.p. and estimate the relative accuracy with which it approximates the the exact value of $f(x)$ at that point.

(3 marks)

3. (a) The Fourier transform $\tilde{f}(w)$ of a function $f(t)$ is defined by

$$\tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt}\, \mathrm{d}t.$$

Give the formula, by which $f(t)$ can be found if its Fourier transform $\tilde{f}(w)$ is known.

(6 marks)

(b) Show that if $f(t)$ is given by

$$f(t) = \begin{cases} 1, & 0 < t < 2, \\ 0, & \text{otherwise}, \end{cases}$$

its Fourier transform is

$$\tilde{f}(w) = \frac{i}{w\sqrt{2\pi}} \left(e^{-2iw} - 1\right)$$

(13 marks)

(c) Using the result of parts (a) and (b), write down the integral which represents $f(t)$ defined as above.

By putting $t = 1$ in this result, evaluate

$$\int_{0}^{\infty} \frac{\sin w}{w}\, \mathrm{d}w.$$

(6 marks)

4. (a) Find the function of t whose Laplace transform is:

$$\frac{1}{(s-3)(s+2)}$$

(4 marks)

(b) Find the function of t whose Laplace transform is:

$$\frac{s-1}{s^2+2s+5}$$

(8 marks)

(c) Find, **using the Laplace Transform**, the solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4y = \cos(2t),$$

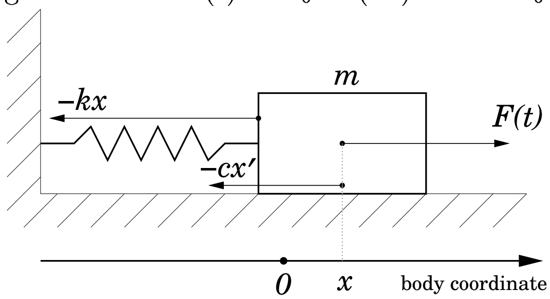
which satisfies initial conditions

$$y(0) = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t}(0) = 1.$$

(13 marks)

Note: a table of standard Laplace transforms is available on page 7.

5. A body of mass $m = 1\,\mathrm{kg}$ is placed on a horizontal surface and attached to a spring with the Hooke's spring constant of $k = 1\,\mathrm{N/m}$ (see the diagram). The horizontal coordinate x of the body is measured with respect to the equilibrium position of the spring. The frictional force acting on the body is proportional to its velocity with the coefficient $c = 2\,\mathrm{Ns/m}$. The external horizontal force applied to the body is changing with time according to the law $F(t) = F_0 \sin(\omega t)$ where $F_0 = 25\,\mathrm{N}$ and $\omega = 2\,\mathrm{s}^{-1}$.



(a) Write down a differential equation for the coordinate $x(t)$ of the body.
(5 marks)

(b) Using the Laplace transform, or otherwise, find the solution to this equation for the initial conditions $x(0) = 0$, $\mathrm{d}x/\mathrm{d}t(0) = 0$.
(15 marks)

Note: a table of standard Laplace transforms is available on page 7.

(c) Represent this solution as a sum of the free movement and the forced oscillations. Based on the form of the free movement, or otherwise, classify this system as underdamped, critically damped or overdamped. Find the coordinate x of the body at the time $t = 10\pi\,\mathrm{s}$, with the precision of 5 significant figures.
(5 marks)

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Table of Laplace transforms

$f(t)$ (original)	$F(s)$ (image)	$f(t)$ (original)	$F(s)$ (image)
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
t	$\frac{1}{s^2}$	te^{at}	$\frac{1}{(s-a)^2}$
t^2	$\frac{2}{s^3}$	t^2e^{at}	$\frac{2}{(s-a)^3}$
t^n	$\frac{n!}{s^{n+1}}$	t^ne^{at}	$\frac{n!}{(s-a)^{n+1}}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$\frac{t\sin(\omega t)}{2\omega}$	$\frac{s}{(s^2+\omega^2)^2}$	$\frac{t\sin(\omega t)}{2\omega}e^{at}$	$\frac{s-a}{((s-a)^2+\omega^2)^2}$
$\frac{\sin(\omega t)-\omega t\cos(\omega t)}{2\omega^2}$	$\frac{\omega}{(s^2+\omega^2)^2}$	$\frac{\sin(\omega t)-\omega t\cos(\omega t)}{2\omega^2}e^{at}$	$\frac{\omega}{((s-a)^2+\omega^2)^2}$
$y(t)$	$Y(s)$	$e^{at}y(t)$	$Y(s-a)$
$\frac{\mathrm{d}y(t)}{\mathrm{d}t}$	$sY(s)-y(0)$	$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}$	$s^2Y(s)-sy(0)-y'(0)$