MATH264 January 2007 Exam Solutions

All questions similar to seen exercises except where marked as Bookwork (B) or Unseen (U).

1. (a) Write $X = X_1 + \cdots + X_n$, where $X_1, \ldots, X_n$ are independent Bernoulli random variables each with success probability p.
That is, $P(X_i = 0) = 1 - p$, $P(X_i = 1) = p$.
Then $E[X_i] = (1-p) \times 0 + p \times 1 = p$,
$\mathrm{Var}[X_i] = (1-p) \times 0^2 + p \times 1^2 - p^2 = p - p^2 = p(1-p)$.
Hence

$$\begin{aligned} E[X] &= E[X_1 + \cdots + X_n] = E]X_1] + \cdots + E[X_n] = p + \cdots + p = np, \\ \mathrm{Var}[X] &= \mathrm{Var}[X_1 + \cdots + X_n] = \mathrm{Var}[X_1] + \cdots + \mathrm{Var}[X_n] = np(1-p). \end{aligned}$$

(b) $E[Y_i] = \left(1 - \left(\frac{1}{2}\right)^i\right) \times 0 + \left(\frac{1}{2}\right)^i \times 1 = \left(\frac{1}{2}\right)^i$

$$\mathrm{Var}[Y_i] = \left(1 - \left(\frac{1}{2}\right)^i\right) \times 0^2 + \left(\frac{1}{2}\right)^i \times 1^2 - \left(\frac{1}{2}\right)^{2i} = \left(\frac{1}{2}\right)^i - \left(\frac{1}{2}\right)^{2i} = \left(\frac{1}{2}\right)^i \left(1 - \left(\frac{1}{2}\right)^i\right)$$

(c) (i) Probability mass function of S_3:

$$\begin{aligned} P(S_3 = 0) &= P(Y_1 = Y_2 = Y_3 = 0) = \frac{1}{2} \times \frac{3}{4} \times \frac{7}{8} = \frac{21}{64} \\ P(S_3 = 1) &= P(Y_1 = 1, Y_2 = Y_3 = 0) + P(Y_2 = 1, Y_1 = Y_3 = 0) \\ &\quad + P(Y_3 = 1, Y_1 = Y_2 = 0) \\ &= \frac{1}{2} \times \frac{3}{4} \times \frac{7}{8} + \frac{1}{4} \times \frac{1}{2} \times \frac{7}{8} + \frac{1}{8} \times \frac{1}{2} \times \frac{3}{4} = \frac{21+7+3}{64} = \frac{31}{64} \\ P(S_3 = 2) &= P(Y_1 = Y_2 = 1, Y_3 = 0) + P(Y_2 = Y_3 = 1, Y_1 = 0) \\ &\quad + P(Y_3 = Y_1 = 1, Y_2 = 0) \\ &= \frac{1}{2} \times \frac{1}{4} \times \frac{7}{8} + \frac{1}{4} \times \frac{1}{8} \times \frac{1}{2} + \frac{1}{8} \times \frac{1}{2} \times \frac{3}{4} = \frac{7+1+3}{64} = \frac{11}{64} \\ P(S_3 = 3) &= P(Y_1 = Y_2 = Y_3 = 1) = \frac{1}{2} \times \frac{1}{4} \times \frac{1}{8} = \frac{1}{64} \end{aligned}$$

(ii)

$$\begin{aligned} E[S_n] &= \sum_{i=1}^n E[Y_i] = \sum_{i=1}^n \left(\frac{1}{2}\right)^i = \frac{1 - (1/2)^{n+1}}{1 - (1/2)} - 1 = 2\left(1 - (1/2)^{n+1}\right) - 1 \\ &= 1 - (1/2)^n \\ \mathrm{Var}[S_n] &= \sum_{i=1}^n \mathrm{Var}[Y_i] = \sum_{i=1}^n \left(\frac{1}{2}\right)^i - \sum_{i=1}^n \left(\frac{1}{4}\right)^i = 1 - (1/2)^n - \left(\frac{1 - (1/4)^{n+1}}{1 - (1/4)} - 1\right) \\ &= 2 - (1/2)^n - (4/3)\left(1 - (1/4)^{n+1}\right) = (2/3) - (1/2)^n + (1/3)(1/4)^n \end{aligned}$$

(iii) $E[\bar{Y}] = E[S_n/n] = E[S_n]/n = \left(1 - (1/2)^n\right)/n \to 0$ as $n \to \infty$.
This seems intuitively reasonable because $P(Y_i = 0) \to 1$ as $n \to \infty$, so expect only finitely many of the Y_i to be non-zero. (NB: Borel-Cantelli lemmas not covered in this module.)

2. (a) Memoryless property: for $t, s > 0$, $P(T > t + s \mid T > t) = P(T > s)$.

Intuitively, this means that knowledge that an item whose lifetime is distributed as T has already survived for time t does not alter the distribution of the remaining lifetime from t onwards.

For exponential distribution,

$$\begin{aligned} P(T > t + s \mid T > t) &= \frac{P(T > t + s \text{ and } T > t)}{P(T > t)} = \frac{P(T > t + s)}{P(T > t)} = \frac{\mathrm{e}^{-\lambda(t+s)}}{\mathrm{e}^{-\lambda t}} = \mathrm{e}^{-\lambda s} \\ &= P(T > s), \text{ as required.} \end{aligned}$$

(b) Weibull density: For $x \geq 0$,

$$\begin{aligned} f_X(x) &= \frac{d}{dx}\left(1 - \exp\left\{-\left(\frac{x}{\theta}\right)^\beta\right\}\right) = \beta\left(\frac{x}{\theta}\right)^{\beta-1}\left(\frac{1}{\theta}\right)\exp\left\{-\left(\frac{x}{\theta}\right)^\beta\right\} \\ &= \frac{\beta}{\theta^\beta}\, x^{\beta-1}\exp\left\{-\left(\frac{x}{\theta}\right)^\beta\right\} \end{aligned}$$

with $f_X(x) = 0$ for $x < 0$.

For Weibull distribution,

$$P(X > a + b \mid X > a) = \frac{P(X > a + b)}{P(X > a)} = \frac{\exp\left\{-\left(\frac{a+b}{\theta}\right)^\beta\right\}}{\exp\left\{-\left(\frac{a}{\theta}\right)^\beta\right\}} = \exp\left\{\frac{a^\beta - (a+b)^\beta}{\theta^\beta}\right\}$$

In the case $\beta = 2$,

$$\frac{P(X > a + b \mid X > a)}{P(X > b)} = \frac{\exp\left\{\frac{a^2 - (a+b)^2}{\theta^2}\right\}}{\exp\left\{-\left(\frac{b}{\theta}\right)^2\right\}} = \exp\left\{\frac{a^2 + b^2 - (a+b)^2}{\theta^2}\right\} = \exp\left\{-\frac{2ab}{\theta^2}\right\}$$

Ratio is not equal to 1, so distribution does not possess memoryless property.

Weibull distribution does possess the memoryless property when $\beta = 1$.

(c) For $\beta = 2$, $\theta = 1$, have

$$g(a) = \exp\left\{a^2 - (a+b)^2\right\} = \exp\left\{-2ab - b^2\right\}$$

Graph:

Interpretation: Graph is decreasing in a, so the older the component is, the lower the probability that it will survive for a further time b, whatever the value of $b > 0$.

3. (a) For g strictly increasing,

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(g(X) \le y) = P\left(X \le g^{-1}(y)\right) = F_X\left(g^{-1}(y)\right) \\ \Rightarrow f_Y(y) &= f_X\left(g^{-1}(y)\right) \frac{d}{dy} g^{-1}(y) \end{aligned}$$

For g strictly decreasing,

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(g(X) \le y) = P\left(X \ge g^{-1}(y)\right) = 1 - F_X\left(g^{-1}(y)\right) \\ \Rightarrow f_Y(y) &= -f_X\left(g^{-1}(y)\right) \frac{d}{dy} g^{-1}(y) \end{aligned}$$

In either case,

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \left| \frac{d}{dy} g^{-1}(y) \right| .$$

(b) (i) For $0 < x < 2$,

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \int_0^x \left(u^3/4\right)\,du = \left[u^4/16\right]_0^x = x^4/16$$

So in full,

$$F(x) = \begin{cases} 0 & x \le 0 \\ x^4/16 & 0 < x < 2 \\ 1 & x \ge 2 \end{cases}$$

(ii) With $Y = \sqrt{X/2}$, then for $y > 0$,

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(\sqrt{X/2} \le y) = P(X \le 2y^2) = F_X(2y^2) \\ \text{so that } F_Y(y) &= \begin{cases} 0 & y \le 0 \\ y^8 & 0 < y < 1 \\ 1 & y \ge 1 \end{cases} \end{aligned}$$

Differentiating,

$$f_Y(y) = \begin{cases} 8y^7 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(Alternatively, use formula from part (a) for f_Y and then integrate to find F_Y.) Range is $0 < Y < 1$.

$$E[Y^n] = \int_0^1 y^n \times 8y^7\,dy = \left[8y^{n+8}/(n+8)\right]_0^1 = 8/(n+8)$$

Hence $E[Y] = 8/9$ and $\mathrm{Var}[Y] = (8/10) - (8/9)^2 = 4/405 \approx 0.00988$.

4. (a) Region of non-zero density:

(b) Marginal density:

$$f_Y(y) \;=\; \int_{x=-y}^{y} \frac{\mathrm{e}^{-y}}{2y}\,dx = \left[\frac{x\mathrm{e}^{-y}}{2y}\right]_{x=-y}^{y} = \frac{y\mathrm{e}^{-y}}{2y} - \frac{-y\mathrm{e}^{-y}}{2y} = \mathrm{e}^{-y} \text{ for } y > 0$$

Conditional density:

$$f_{X|Y}(x \mid y) \;=\; \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\mathrm{e}^{-y}/2y}{\mathrm{e}^{-y}} = \frac{1}{2y} \text{ for } -y < x < y$$

Distribution of Y is $\exp(1)$; conditional distribution of X is Uniform$(-y, y)$.

(c) Expectations:

$$\begin{aligned}
E[X] &= \int_{y=0}^{\infty}\int_{x=-y}^{y} x\,\frac{\mathrm{e}^{-y}}{2y}\,dx\,dy = \int_{y=0}^{\infty}\left[\frac{x^2\mathrm{e}^{-y}}{4y}\right]_{x=-y}^{y} dy \\
&= \int_{y=0}^{\infty}\left(\frac{y^2\mathrm{e}^{-y}}{2y} - \frac{y^2\mathrm{e}^{-y}}{2y}\right) dy = 0 \\
E[Y] &= \int_{y=0}^{\infty} y f_Y(y)\,dy = \int_{y=0}^{\infty} y\mathrm{e}^{-y}\,dy \\
&= \left[-y\mathrm{e}^{-y}\right]_{y=0}^{\infty} + \int_{y=0}^{\infty} \mathrm{e}^{-y}\,dy = 0 + \left[-\mathrm{e}^{-y}\right]_{y=0}^{\infty} = 1
\end{aligned}$$

(d) Covariance:

$$\begin{aligned}
E[XY] &= \int_{y=0}^{\infty}\int_{x=-y}^{y} xy\,\frac{\mathrm{e}^{-y}}{2y}\,dx\,dy = \int_{y=0}^{\infty}\left[\frac{x^2\mathrm{e}^{-y}}{4}\right]_{x=-y}^{y} dy \\
&= \int_{y=0}^{\infty}\left(\frac{y^2\mathrm{e}^{-y}}{4} - \frac{y^2\mathrm{e}^{-y}}{4}\right) dy = 0
\end{aligned}$$

so that $\mathrm{Cov}[X,Y] = E[XY] - E[X]E[Y] = 0 - 0 \times 1 = 0$.

(e) X and Y are not independent; the range of possible X values depends upon the value of Y.

5. (a) $U = 2X + Y, V = 3X/Y$, so

$$\begin{aligned} X &= \frac{YV}{3} \Rightarrow U = \frac{2YV}{3} + Y = Y\left(\frac{2V}{3}+1\right) = \frac{Y}{3}(2V+3) \Rightarrow Y = \frac{3U}{2V+3} \\ \text{and so } X &= \frac{YV}{3} = \left(\frac{3U}{2V+3}\right)\left(\frac{V}{3}\right) = \frac{UV}{2V+3} \end{aligned}$$

Differentials:

$$\begin{aligned} \frac{\partial x}{\partial u} &= \frac{v}{2v+3} & \frac{\partial x}{\partial v} &= \frac{u\left((2v+3)\times 1 - v \times 2\right)}{(2v+3)^2} = \frac{3u}{(2v+3)^2} \\ \frac{\partial y}{\partial u} &= \frac{3}{2v+3} & \frac{\partial y}{\partial v} &= \frac{(2v+3)\times 0 - 3u \times 2}{(2v+3)^2} = \frac{-6u}{(2v+3)^2} \end{aligned}$$

Jacobian:

$$J = \left(\frac{v}{2v+3}\right)\left(\frac{-6u}{(2v+3)^2}\right) - \left(\frac{3}{2v+3}\right)\left(\frac{3u}{(2v+3)^2}\right) = \frac{-6uv-9u}{(2v+3)^3} = \frac{-3u}{(2v+3)^2}$$

(b) Density:

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \\ &= (2/\pi)\exp\left\{-\left(\left(\frac{uv}{2v+3}\right)^2 + \left(\frac{3u}{2v+3}\right)^2\right)\Big/ 2\right\}\left|\frac{-3u}{(2v+3)^2}\right| \\ &= \frac{6u}{\pi(2v+3)^2}\exp\left\{-\left(\frac{u^2\left(v^2+9\right)}{2(2v+3)^2}\right)\right\} \qquad u, v > 0 \end{aligned}$$

(c) Marginal density of V:

$$\begin{aligned} f_V(v) &= \frac{6}{\pi(2v+3)^2}\int_{u=0}^{\infty} u \exp\left\{-\left(\frac{u^2\left(v^2+9\right)}{2(2v+3)^2}\right)\right\} du \\ &= \frac{6}{\pi(2v+3)^2}\int_{u=0}^{\infty} u \exp\left\{-Au^2\right\} du \qquad \text{with } A = \frac{v^2+9}{2(2v+3)^2} \\ &= \frac{6}{\pi(2v+3)^2}\frac{1}{2A} = \frac{6}{\pi(2v+3)^2}\frac{(2v+3)^2}{v^2+9} = \frac{6}{\pi(v^2+9)} \qquad v > 0 \end{aligned}$$

6. (a) Differentiating,

$$M_X(t) = E[\mathrm{e}^{tX}] \qquad M'_X(t) = E[X\mathrm{e}^{tX}] \qquad M''_X(t) = E[X^2\mathrm{e}^{tX}]$$
$$\Rightarrow \quad M'_X(0) = E[X] \qquad M''_X(0) = E[X^2]$$

and so $\mathrm{Var}[X] = E[X^2] - (E[X])^2 = M''_X(0) - \left(M'_X(0)\right)^2$, as required.

Repeated differentiation similarly gives $M_X^{(n)}(0) = E[X^n]$, where $M_X^{(n)}$ denotes the nth derivative of M_X.

With $Y = a + bX$,

$$M_Y(t) = E[\mathrm{e}^{tY}] = E[\mathrm{e}^{t(a+bX)}] = E[\mathrm{e}^{at} \times \mathrm{e}^{tbX}] = \mathrm{e}^{at} M_X(bt)$$

(b) With $Z \sim N(0,1)$,

$$\begin{aligned} M_Z(t) &= E[\mathrm{e}^{tZ}] = \int_{-\infty}^{\infty} \mathrm{e}^{tz} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{tz-(z^2/2)}\, dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-((z-t)^2/2)+(t^2/2)}\, dz = \frac{\mathrm{e}^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-(z-t)^2/2}\, dz \end{aligned}$$

Substituting $u = z - t$, so that $du = dz$, then

$$M_Z(t) = \frac{\mathrm{e}^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/2}\, du = \mathrm{e}^{t^2/2} \int_{-\infty}^{\infty} f_Z(u)\, du = \mathrm{e}^{t^2/2}$$

since the standard normal density f_Z integrates to 1.

Now for $X = \mu + \sigma Z$, have $M_X(t) = \mathrm{e}^{\mu t} M_Z(\sigma t) = \mathrm{e}^{\mu t}\mathrm{e}^{(\sigma t)^2/2} = \mathrm{e}^{\mu t + (\sigma^2/2)t^2}$.

Cumulant generating function $K_X(t) = \ln\left(M_X(t)\right) = \mu t + (\sigma^2/2)t^2$.

The cumulants $\kappa_1, \kappa_2, \ldots$ are defined to be the coefficients in the power series $K_X(t) = \kappa_1 t + \kappa_2 \frac{t^2}{2!} + \kappa_3 \frac{t^3}{3!} + \cdots$, so in the case of the normal distribution we see that the first cumulant is equal to the mean, the second cumulant is equal to the variance, and all higher cumulants are zero.

In general, first cumulant is the mean, second cumulant is variance, higher cumulants not generally zero, for non-normal distributions.

7. (a) Suppose $X_1, X_2, \ldots$ are independent, identically distributed random variables with mean μ and variance σ^2. Then for any real number x,

$$P\left(\frac{\sum_{i=1}^{n}(X_i - \mu)}{\sigma\sqrt{n}} \leq x\right) \quad \rightarrow \quad \Phi(x) \text{ as } n \rightarrow \infty$$

where $\Phi(x)$ is the standard normal distribution function.

In practice, approximation of the sample mean $\bar{X}$ is better (a) for larger values of n; (b) when the distribution of X_i is reasonably symmetrical.

(b) (i) For V_i uniformly distributed on $[0,1]$ have $E[V_i] = 1/2$, $\text{Var}[V_i] = 1/12$.

$$\begin{aligned} E[\bar{V}] &= E[(V_1 + \cdots + V_n)/n] = (E[V_1] + \cdots + E[V_n])/n = ((1/2)n)/n = 1/2 \\ \text{Var}[\bar{V}] &= \text{Var}[(V_1 + \cdots + V_n)/n] = (\text{Var}[V_1] + \cdots + \text{Var}[V_n])/n^2 \\ &= ((1/12)n)/n^2 = 1/(12n) \end{aligned}$$

(ii) With $n = 12$,

$$\begin{aligned} P(\bar{V} < 0.6) &\approx P\left(Z < \frac{0.6 - (1/2)}{\sqrt{1/144}}\right) \text{ where } Z \sim N(0,1) \\ &= P(Z < 1.2) = \Phi(1.2) = 0.8849 \end{aligned}$$

(iii) Value of $n = 12$ is not very large, but since the distribution of V_i is perfectly symmetrical might still expect CLT to perform reasonably well. Range of $\bar{V}$ is $[0,1]$, whereas range of the normal approximation is $(-\infty, \infty)$, which would seem to be a deficiency in the approximation; however, $\bar{V}$ has mean $1/2$ and standard deviation $1/12$, so that six standard deviations either side of the mean lie within the interval $[0,1]$, and hence the probability which the normal approximation assigns outside this interval is negligible.

(iv) Require

$$\begin{aligned} P(\bar{V} < 0.6) &> 0.975 \\ \text{Approximately equivalent to } P\left(Z < \frac{0.6 - (1/2)}{\sqrt{1/(12n)}}\right) &> 0.975 \\ P\left(Z < 0.1\sqrt{12n}\right) &> 0.975 \\ 0.1\sqrt{12n} &> 1.96 \\ n &> 19.6^2/12 = 32.013 \end{aligned}$$

So required value is $n = 33$.