1. (a) Let X be binomially distributed with parameters (n, p). By representing X as a sum of independent Bernoulli random variables, derive expressions for $E[X]$ and $\text{Var}[X]$. [4 marks]

(b) Suppose $Y_1, Y_2, Y_3, \ldots$ are independent Bernoulli random variables with respective success probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$. That is, for $i = 1, 2, \ldots$ the probability mass function of Y_i is given by

$$\begin{aligned} P(Y_i = 0) &= 1 - \left(\frac{1}{2}\right)^i, \\ P(Y_i = 1) &= \left(\frac{1}{2}\right)^i. \end{aligned}$$

Find $E[Y_i]$ and $\text{Var}[Y_i]$. [2 marks]

(c) For $n = 1, 2, \ldots$ define $S_n = Y_1 + \cdots + Y_n$, where $Y_1, Y_2, \ldots$ are as in part (b) above.

(i) For $n = 3$, show that the probability mass function of S_3 is given by

$$\begin{aligned} P(S_3 = 0) &= 21/64, \\ P(S_3 = 1) &= 31/64, \\ P(S_3 = 2) &= 11/64, \\ P(S_3 = 3) &= 1/64. \end{aligned}$$

[6 marks]

(ii) Find $E[S_n]$ and $\text{Var}[S_n]$.

[You may use without proof the result that

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}.$$]

[6 marks]

(iii) Defining the sample mean $\bar{Y}_n = S_n/n$, show that $\lim_{n\to\infty} E\left[\bar{Y}_n\right] = 0$. Comment on why this result is intuitively reasonable. [2 marks]

2. (a) Suppose the random variable T is exponentially distributed with parameter λ, so that T has distribution function

$$F_T(t) \quad = \quad \begin{cases} 0 & t < 0, \\ 1 - \mathrm{e}^{-\lambda t} & t \geq 0. \end{cases}$$

Define the *memoryless property*, explain the intuitive meaning of this property, and show that the distribution of T has the memoryless property. [4 marks]

(b) The random variable X is said to follow the Weibull distribution with parameters $\beta > 0$, $\theta > 0$ if the distribution function of X is given by

$$F_X(x) \quad = \quad \begin{cases} 0 & x < 0, \\ 1 - \exp\left\{ -\left(\frac{x}{\theta}\right)^{\beta} \right\} & x \geq 0. \end{cases}$$

Derive an expression for the probability density function $f_X(x)$ of X. [3 marks]

Suppose the random variable X follows the Weibull distribution with parameters β, θ. For $a, b > 0$ find an expression for $P\left(X > a + b \mid X > a\right)$. [4 marks]

By considering the ratio

$$\frac{P\left(X > a + b \mid X > a\right)}{P(X > b)}$$

show that in the case $\beta = 2$, the distribution of X does not possess the memoryless property. [4 marks]

For which values of the parameters β, θ, if any, does the Weibull distribution possess the memoryless property? [1 mark]

(c) Suppose that the lifetime Y (in years) of a particular electrical component follows the Weibull distribution with parameters $\beta = 2$, $\theta = 1$. Treating b as a fixed constant, sketch a graph of the function $g(a)$ defined by

$$g(a) \quad = \quad P\left(Y > a + b \mid Y > a\right).$$

Give your intepretation of the shape of the graph of $g(a)$. [4 marks]

3. (a) Suppose that X is a continuous random variable with probability density function $f_X(x)$, and that the random variable Y is defined by $Y = g(X)$ for some strictly monotonic function $g(x)$. Show that the probability density function of Y is given by

$$f_Y(y) \quad = \quad f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right|.$$

[5 marks]

(b) Suppose X is a continuous random variable with density

$$f(x) \quad = \quad \begin{cases} x^3/4 & 0 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the cumulative distribution function $F(x)$ of X. [3 marks]

(ii) Find the cumulative distribution function and the probability density function of the random variable Y defined by $Y = \sqrt{X/2}$. What is the range of Y?

[8 marks]

Find an expression for $E\left[Y^n\right]$ where $n = 0, 1, 2, \ldots$ Hence write down the values of $E[Y]$ and $\text{Var}[Y]$. [4 marks]

4. Suppose that X, Y are continuous random variables with joint density function

$$f_{X,Y}(x, y) \quad = \quad \frac{e^{-y}}{2y}, \qquad\qquad y > 0, \quad -y < x < y.$$

(a) Draw the region of non-zero density. [3 marks]

(b) Find the marginal density $f_Y(y)$ of Y and the conditional density $f_{X|Y}(x \mid y)$ of X given $Y = y$ for $y > 0$. [6 marks]

For each of these two distributions, give the standard name of the distribution, specifying any parameter values. [2 marks]

(c) Find $E[X]$ and $E[Y]$. [3 marks]

(d) Find $E[XY]$ and hence find the covariance $\text{Cov}[X, Y]$. [4 marks]

(e) Are X and Y independent? Explain your answer. [2 marks]

5. Suppose that X, Y are continuous random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} (2/\pi)\exp\left\{-\left(x^2+y^2\right)/2\right\} & \text{for } x, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define the random variables U, V by

$$U = 2X + Y, \qquad V = 3X/Y.$$

(a) Show that the Jacobian, J, is given by

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{-3u}{(2v+3)^2}.$$

[6 marks]

(b) Find the joint density $f_{U,V}(u,v)$. [7 marks]

(c) Find the marginal density of V. [7 marks]

[For part (c) you may use without proof the result that for $A > 0$,

$$\int_0^\infty u e^{-Au^2}\, du = \frac{1}{2A}. \qquad]$$

6. (a) For any random variable X the moment generating function $M_X(t)$ is defined by $M_X(t) = E\left[e^{tX}\right]$.

Show that (i) $E[X] = M'_X(0)$ and (ii) $\mathrm{Var}[X] = M''_X(0) - (M'_X(0))^2$, where $M'_X(t)$ and $M''_X(t)$ denote the first and second derivatives, respectively, of $M_X(t)$ with respect to t. [3 marks]

Write down an expression for $E[X^n]$, where $n = 1, 2, \ldots$, in terms of derivatives of the moment generating function $M_X(t)$. [1 mark]

Defining the random variable Y to be $Y = a + bX$, where a, b are non-random, show that the moment generating function of Y is given by

$$M_Y(t) \quad = \quad e^{at} M_X(bt).$$

[2 marks]

(b) Suppose Z follows the standard normal distribution, $Z \sim N(0,1)$, so that Z has probability density function

$$f_Z(z) \quad = \quad \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2} \qquad\qquad -\infty < z < \infty.$$

Show that the moment generating function of Z is given by $M_Z(t) = e^{t^2/2}$. [8 marks]

Hence derive an expression for the moment generating function $M_X(t)$ of the normal random variable X with mean μ and variance σ^2 given by $X = \mu + \sigma Z$, and show that the cumulant generating function $K_X(t) = \ln(M_X(t))$ of X is given by

$$K_X(t) = \mu t + \left(\sigma^2/2\right) t^2.$$

[3 marks]

Based on this expression for $K_X(t)$, what can be said about the first, second, and higher cumulants of the $N(\mu, \sigma^2)$ distribution? To what extent do these statements generalise to distributions other than the normal distribution?

[3 marks]

7. (a) Give a careful statement of the Central Limit Theorem. [5 marks]

When using the Central Limit Theorem to approximate the distribution of the sample mean $\bar{X}$, what factors affect the accuracy of the approximation in practice? [2 marks]

(b) Suppose $V_1, V_2, \ldots, V_n$ are independent, identically distributed random variables, each being uniformly distributed on the interval $[0,1]$, and define $\bar{V} = (V_1 + \cdots + V_n)/n$.

(i) Show that $E\left[\bar{V}\right] = 1/2$ and $\text{Var}\left[\bar{V}\right] = \dfrac{1}{12n}$. [3 marks]

(ii) For the case $n = 12$, use the Central Limit Theorem to approximate $P\left(\bar{V} < 0.6\right)$. [2 marks]

(iii) Comment on whether you would expect the Central Limit Theorem to provide a good approximation for the probabiilty computed in part (ii). In your answer you should consider the range of $\bar{V}$, as well as any factors mentioned in your answer to part (a) above. [4 marks]

(iv) Use the Central Limit Theorem to find (approximately) the smallest n for which $P\left(\bar{V} < 0.6\right) > 0.975$. [4 marks]