1. In an automatic telephone exchange the probability that any one call is wrongly connected is 0.001.

(a) For a day when 2000 independent calls are connected, determine the probability that at most 2 wrong connections are made using the exact formula. (The answer must be presented with precision to 6 decimal places.) [7 marks]

(b) Use the Poisson distribution to find an approximate value of the probability asked for in part (a), with precision to 6 decimal places. Comment on the accuracy of the Poisson approximation. [4 marks]

(c) What is the minimum number of independent calls required before there is a probability of 0.9 that at least one of the calls is wrongly connected. [9 marks]

2. An expensive piece of equipment is insured with the insurance company *First Pacific Inc.* according to the following table in which T denotes the first time (in years) that the equipment fails to function within specified parameters; X denotes the amount (in millions of pounds) *First Pacific Inc.* will have to pay to the owners of the equipment:

$$\begin{array}{lll} X = 5 & \text{if} & T < 1 \\ X = 4 & \text{if} & 1 \le T < 2 \\ X = 2 & \text{if} & 2 \le T < 3 \\ X = 0 & \text{if} & T \ge 3 \end{array}$$

Suppose T is an exponential random variable defined by

$$P(T > t) = \exp(-0.5t), \ t \ge 0.$$

(a) Find the probability that *First Pacific Inc.* will pay compensation higher than £3 million. [4 marks]

(b) Find the expected compensation *First Pacific Inc.* will pay. [10 marks]

(c) Suppose *First Pacific Inc.* buys a policy from *Lloyds of London PLC.* to protect itself against very large claims. According to this policy, if *First Pacific Inc.* agrees to settle a claim of £x million, it will, in fact, pay only £$f(x)$ million, where $f(x) \le x$, and the rest will be paid by *Lloyds of London PLC.*. Suppose that

$$f(x) = x \ \text{ if } \ x < 3, \quad f(x) = 3 \ \text{ if } \ x \ge 3.$$

Find the expected compensation *First Pacific Inc.* will pay assuming it has purchased the policy described in this part of the question. [6 marks]

3. Let $X \sim U[0,4]$ be a uniform on $[0,4]$ random variable and $Y = \sqrt{X}$.

(a) Write down the probability density function of X. [2 marks]
(b) Determine the range of Y. [2 marks]
(c) Find the cumulative distribution function of Y. [8 marks]
(d) Find the probability density function of Y. [4 marks]
(e) Find the mathematical expectation of Y. [4 marks]

4. Suppose the random variables X and Y are jointly continuous with joint density function

$$f(x,y) = 2(x+y), \quad 0 \le y \le x \le 1.$$

(a) Draw the region where the density is positive. [3 marks]
(b) Find the marginal densities of X and Y. [9 marks]
(c) Find the conditional density of Y given $X = x,\ 0 \le x \le 1$. [3 marks]
(d) Calculate $P(Y > \frac{1}{3} | X = 1)$. [5 marks]

5. Suppose the random variables X and Y are jointly continuous with joint density function

$$f(x,y) = xe^{-x(y+1)}, \qquad x, y \ge 0.$$

Let the random variables U and V be defined by

$$U = X + Y, \quad V = \ln(Y+1).$$

Find the joint density of U and V, and indicate the range of the random vector (U,V). [20 marks]

6. The random variables $X_1, \ldots, X_{100}$ are independent and identically distributed, each with probability mass function

$$p(-2) = 0.5, \quad p(0) = 0.4, \quad p(1) = 0.1.$$

(a) Using the Central Limit Theorem, find approximations for
(i) $P(\sum_{i=1}^{100} X_i \le -70)$ [6 marks]
(ii) $P(\sum_{i=1}^{100} X_i^2 \ge 200)$ [6 marks]

(b) Consider now a sample $X_1, \ldots, X_n$, of size n. How large must n be to ensure that the sum of squares, $\sum_{i=1}^{100} X_i^2$, is greater than 200 with probability greater than 0.99? [8 marks]

7. Suppose the random variable X has a χ^2_n distribution, i.e. it has density function

$$f_X(x) = \frac{1}{2^{n/2}\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-x/2}, \quad x > 0.$$

(a) Show that the random variable $U = \sqrt{X/n}$ has density function

$$f_U(u) = \frac{1}{2^{(n/2)-1}\Gamma(\frac{n}{2})} n^{(n/2)} (u^2)^{\frac{n-1}{2}} e^{-nu^2/2}.$$

[7 marks]

(b) Suppose Y is a standard normal random variable which is independent of X. Using the result of part (a) show that the t–statistic

$$T = \frac{Y}{\sqrt{X/n}}$$

has density function

$$f_T(t) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})} \left(1 + \frac{t^2}{n}\right)^{-\frac{(n+1)}{2}}.$$

[13 marks]

Hint: Recall that the quotient Y/U of independent continuous random variables Y and U has density function

$$f(t) = \int_{-\infty}^{\infty} |u| f_U(u) f_Y(ut) du$$

and that the equality

$$\int_0^{\infty} x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

holds for all $\alpha, \lambda > 0$.

[NORMAL DISTRIBUTION TABLE]