1. (a) Write down the distribution of a Poisson random variable with parameter λ and give an interpretation of λ in terms of the number of events per unit time. Explain when a Poisson random variable is a good approximation to a binomial random variable and state the relationship between the parameters of these two distributions. [6 marks]

(b) A team of astronomers is studying a sector of the sky containing 53,294 stars visible with their telescope. In the 972 days since their research began, they have observed 8 supernova explosions.

(i) Find the probability that at least one supernova explosion will be observed by the team in the next 92 days. [7 marks]

(ii) The team wants to observe another supernova explosion with probability at least 0.95. For how many more days should their research continue? [7 marks]

2. (a) A company is considering developing and marketing a new product. It is estimated to be twice as likely that the product would prove to be successful as unsuccessful. If it were successful, the expected profit would be £1,500,000. If unsuccessful, the expected loss would be £1,800,000. Compute the expected profit if the product is developed and marketed. [3 marks]

(b) A marketing survey can be conducted at a cost of £300,000 to predict whether the product would be successful. Past experience with such surveys indicates that successful products have been predicted to be successful 80% of the time, whereas unsuccessful products have been predicted to be unsuccessful 70% of the time. Calculate the posterior distribution of whether the product would be successful if the marketing survey predicts success. [5 marks]

Do the same if the survey predicts that the product would be unsuccessful. [5 marks]

(c) Suppose the company develops the new product if the marketing survey predicts success (and does not develop if the survey predicts that the product would be unsuccessful). Calculate the expected profit in this case. [5 marks]

Is it worth carrying the market survey out? [2 marks]

3. Suppose the random variables X and Y are jointly continuous with density

$$f(x,y) = 60x^2y$$

on the region

$$0 < x, \quad 0 < y, \quad x + y < 1,$$

and zero elsewhere.

(a) Sketch the region where $f(x,y) > 0$. [2 marks]

(b) Find the marginal densities of X and Y. [6 marks]

(c) Find the conditional density of Y given $X = x$. [4 marks]

(d) Find $P(Y > 0.1 | X = 0.5)$. [8 marks]

4. Random variables X and Y are jointly continuous with density

$$f(x,y) = \begin{cases} (2/\pi)\exp\left\{-\left(x^2+y^2\right)/2\right\} & \text{if } x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent? Justify your answer. [4 marks]

(b) Suppose a researcher is interested in the random variables U and V which are defined as follows:

$$U = X + 2Y, \qquad V = X/Y.$$

Find the joint density of U and V. [10 marks]

(c) Find the marginal density of V. [6 marks]

5. Define the moment generating function and state its properties. [4 marks]

Consider a geometric random variable X with probability mass function

$$P(X = k) = pq^{k-1}, \quad k = 1, 2, \ldots,$$

where p and q are positive numbers such that $p + q = 1$.

(a) Show that the moment generating function of X is

$$M(t) = \frac{pe^t}{1 - qe^t}, \quad t < -\ln q.$$

[8 marks]

(b) Using the above result, find the expected value and variance of X. [8 marks]

6. Suppose that $X_1, \ldots, X_n$ are repeated, independent, unbiased measurements of a quantity, μ, and that $\mathrm{Var} X_i = \sigma^2$. The average of the measurements

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is used as an estimate of μ.

(a) Using the Central Limit Theorem, show that

$$P(|\bar{X} - \mu| < c) \approx 2\Phi\left(\frac{c\sqrt{n}}{\sigma}\right) - 1,$$

where $\Phi(x)$ is the standard normal cumulative distribution function. [10 marks]

(b) Suppose $\sigma = 1$ and $c = 0.5$. How large should n be to ensure that

$$P(|\bar{X} - \mu| < c) \geq 0.97?$$

[10 marks]

[*Hint:* use formula above and the normal tables.]

7. Suppose that X and Y are independent random variables having the exponential distribution with parameter $1/2$.

(a) Calculate the density of the ratio $Z = Y/X$. [10 marks]

Hint: if X and Y are independent random variables then the density of the quotient $Z = Y/X$ is

$$f_Z(z) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(zx) dx.$$

(b) If U and V are independent chi-square random variables with m and n degrees of freedom correspondingly then the distribution of $W = \frac{U/m}{V/n}$ is called the 'F-distribution'. The density function of W is

$$f_W(w) = \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\left(\frac{m}{n}\right)^{m/2} w^{\frac{m}{2}-1}\left(1 + \frac{m}{n}w\right)^{-\frac{m+n}{2}}, \quad w > 0.$$

Show that Z from question (a) has F distribution. What are the values m and n for Z? [5 marks]

(c) Show that $E[Z]$ is undefined. [5 marks]

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