MATH264, Summer 2002. **Solutions**

1. **[Similar to homework and examples discussed in class.]**

(a) Let X denote the number of the tickets which go to girls. Then

$$X \sim BIN(4, p),$$

where p is the probability that a girl is randomly selected. This probability is equal to the proportion of girls in class i.e.

$$p = \frac{20}{35} \approx 0.5714.$$

Using the binomial probabilities, we thus get

(i)

$$P(X = 4) = p^4 \approx 0.107$$

(ii)

$$P(X = 2) = \binom{4}{2} p^2 (1 - p)^2 = 6 \left(\frac{20}{35}\right)^2 \left(\frac{15}{35}\right)^2 \approx 0.360.$$

(b) Let X denote the number of individuals with the disease in the group of two thousand. We use the Poisson approximation with

$$\lambda = 2,000 \times 10^{-4} = 0.2.$$

The desired probability is

$$P(x \leq 3) = \sum_{k=0}^{3} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-0.2} \left(1 + 0.2 + \frac{0.2^2}{2} + \frac{0.2^3}{6}\right) \approx 0.9999.$$

Thus we can be practically certain to find 3 or fewer individuals with the disease.

(c) It was appropriate to use the Poisson approximation in part (b) because the number of trials $n = 2,000$ was large, the "probability of success" $p = 10^{-4}$ small and $\lambda = 0.2$ moderate. In part (a), $n = 4$ and $p \approx 0.5714$, so neither n is large, nor p is small.

2. **[All the reasonings are standard, similar to examples discussed in class.]**

Mathematical expectation $E(X)$ exists iff integral $\int_B^\infty \frac{x}{x^3} dx$ is finite. Since

$$\int_B^\infty \frac{dx}{x^2} = -\left.\frac{1}{x}\right|_B^\infty = \frac{1}{B}$$

is finite, $E(X)$ does exist. Similarly,

$$\int_B^\infty \frac{x^2}{x^3} dx = \int_B^\infty \frac{1}{x} dx = \ln x|_B^\infty = +\infty,$$

and $Var(X)$ does not exist.

(a)

$$E(X) = \int_1^\infty x f(x) dx = 4 \int_1^\infty x^{-4} dx = 4/3;$$

$$E(X^2) = \int_1^\infty x^2 f(x) dx = 4 \int_1^\infty x^{-3} dx = 2;$$

$$Var(X) = E(X^2) - (EX)^2 = 2 - \frac{16}{9} = \frac{2}{9};$$

$$\sigma(X) = \frac{\sqrt{2}}{3} \approx 0.471.$$

(b) N=net profit; $N = 0.69\ X$;

$$E[N] = 0.69E[X] = 0.69 \cdot 4/3 = 0.92;$$

$$Var(N) = Var(0.69\ X) = (0.69)^2 Var(X) = (0.69)^2 \frac{2}{9} = 0.106.$$

(c) $Y = N^2$; $E[Y] = E[N^2] = Var(N) + (E[N])^2 = 0.106 + (0.92)^2 = 0.952.$

3. **[Similar to an example presented in class.]**

The transformation considered in the problem is

$$y_1 = x_1 - x_2, \quad y_2 = x_1 + x_2,$$

so the inverse transformation is

$$x_1 = \frac{y_1 + y_2}{2}, \quad x_2 = \frac{y_2 - y_1}{2}.$$

The Jacobian of the inverse transformation thus is

$$J = det \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} = 1/2.$$

Using the formula we have

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}\left(\frac{y_1 + y_2}{2}, \frac{y_2 - y_1}{2}\right)|J|$$

$$= \exp\left\{-\left(\frac{y_1 + y_2}{2} + \frac{y_2 - y_1}{2}\right)\right\}(1/2) = \frac{1}{2}e^{-y_2}.$$

To find the region where this density is positive, we use the conditions

$$\frac{y_1 + y_2}{2} = x_1 > 0, \quad \frac{y_2 - y_1}{2} = x_2 > 0$$

which yield

$$y_2 > -y_1, \quad y_2 > y_1.$$

The region $\Omega = \{(y_1, y_2) : y_2 > -y_1,\ y_2 > y_1\}$ is drawn below.

4. **[Standard, similar to bookwork and homework.]**

(a)

$$\int_0^1 \left[\int_0^\infty (x+y)e^{-x}dx\right] dy = \int_0^1 \left[\int_0^\infty xe^{-x}dx + y\int_0^\infty e^{-x}dx\right] dy$$
$$= \int_0^1 [1+y]dy = [y+\frac{y^2}{2}]_0^1 = \frac{3}{2}.$$

Thus $K = \frac{2}{3}$.

(b)

$$f_X(x) = \int_0^1 \frac{2}{3}(x+y)e^{-x}dy = \frac{2}{3}e^{-x}\left\{x\int_0^1 dy + \int_0^1 ydy\right\} = \frac{2}{3}e^{-x}(x+\frac{1}{2}), \quad x \geq 0.$$

$$f_Y(y) = \int_0^\infty \frac{2}{3}(x+y)e^{-x}dx = \frac{2}{3}\left\{\int_0^\infty xe^{-x}dx + y\int_0^\infty e^{-x}dx\right\} = \frac{2}{3}[1+y], \quad 0 \leq y \leq 1.$$

(c) 6 months= $\frac{1}{2}$ year

$$P(Y \leq \frac{1}{2}) = \int_0^{1/2} \frac{2}{3}(1+y)dy = \frac{2}{3}[y+\frac{y^2}{2}]_0^{1/2} = \frac{2}{3}[1/2+1/8] = \frac{5}{12}.$$

(d)

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{(x+y)e^{-x}}{1+y};$$
$$f_{X|Y}(x|\frac{1}{2}) = \frac{(x+1/2)e^{-x}}{3/2} = \frac{2}{3}(x+\frac{1}{2})e^{-x}.$$

$$E[X|Y=1/2] = \int_0^\infty xf_{X|Y}(x|\frac{1}{2})dx = \frac{2}{3}\int_0^\infty x^2e^{-x}dx + \frac{1}{3}\int_0^\infty xe^{-x}dx = \frac{4}{3}+\frac{1}{3} = \frac{5}{3}.$$

5. **[Bookwork and similar to homework.]**

The covariance of two arbitrary random variables X and Y is

$$\text{Cov}(X,Y) = E\left[(X - EX)(Y - EY)\right].$$

In general,

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y).$$

Summing appropriate rows and columns in the table we obtain

$$P(X=-2) = 0.5,\ \ P(X=2) = 0.5,$$

$$P(Y=-1) = 0.5,\ \ P(Y=1) = 0.5$$

and so

$$E(X) = 0.5\cdot(-2) + 0.5\cdot 2 = 0, \quad E(Y) = 0.5\cdot(-1) + 0.5\cdot 1 = 0.$$

$$\text{Var}(X) = E[X^2] = 4, \quad \text{Var}(Y) = E[Y^2] = 1$$

Using the general formulae above, we get

$$\text{Cov}(X,Y) = E[XY]$$

$$= 0.4(-1)(-2) + 0.1(-2)(1) + 0.1(2)(-1) + 0.4(2)(1) = 1.2.$$

$$\text{Var}(X+Y) = 4 + 1 + 2(1.2) = 7.4.$$

6. **[(a) Bookwork, (b,c) - standard, similar to examples discussed in class]**

(a) The moment generating function of a RV X is defined as

$$M_X(t) = E[e^{tX}].$$

(b)

$$M_X(t) = \int_0^1 e^{tx}dx = \frac{1}{t}[e^t - 1].$$

$$E[X] = M_X'(0) = \lim_{t\to 0}\left[\frac{1-e^t}{t^2} + \frac{e^t}{t}\right] = 1/2.$$

(We use L'Hopital's rule here.)

$$E[X^2] = M_X''(0) = \lim_{t\to 0}\frac{t^3e^t - 2t + 2te^t - 2t^2e^t}{t^4} = 1/3.$$

(We apply L'Hopital's rule twice here.) Therefore, $Var\ X = 1/3 - 1/4 = 1/12$.

The answers obtained by integrating the density function are also OK.

(c) Let $Z = \sum_{i=1}^n X_i$; $a = -\sqrt{3n}$; $b = \frac{2\sqrt{3}}{\sqrt{n}}$. Since X_i are iid,

$$M_Z(t) = [M_X(t)]^n = \frac{1}{t^n}[e^t - 1]^n$$

and

$$M_Y(t) = M_{a+bZ}(t) = e^{-t\sqrt{3n}} \frac{1}{\left[\frac{2t\sqrt{3}}{\sqrt{n}}\right]^n} \left[e^{\frac{2t\sqrt{3}}{\sqrt{n}}} - 1\right]^n.$$

Let us introduce the following denotation

$$u = \frac{t\sqrt{3}}{\sqrt{n}}.$$

Then

$$M_Y(t) = \left(\frac{e^u - e^{-u}}{2u}\right)^{\frac{3t^2}{u^2}}.$$

Now, compute the following limit:

$$\lim_{u\to 0} \frac{1}{u^2} \ln\left(\frac{e^u - e^{-u}}{2u}\right) = \lim_{u\to 0} \frac{\frac{e^u+e^{-u}}{e^u-e^{-u}} - \frac{1}{u}}{2u}$$

$$= \lim_{u\to 0} \frac{e^u - e^{-u}}{4(e^u - e^{-u}) + 2u(e^u + e^{-u})}$$

$$= \lim_{u\to 0} \frac{e^u + e^{-u}}{6(e^u + e^{-u}) + 2u(e^u - e^{-u})} = 1/6.$$

(We have used L'Hopital's rule several times.)

Therefore,

$$\lim_{n\to\infty} M_Y(t) = \lim_{u\to 0} \left(\frac{e^u - e^{-u}}{2u}\right)^{\frac{3t^2}{u^2}} = e^{\frac{1}{6}3t^2} = e^{t^2/2}$$

as we wished to prove.

7. **[Reasonings are standard, but not seen.]**

(a) N_1 follows the binomial distribution $Bin(n,p)$. Since n is large we can use the normal approximation based on the Central Limit Theorem: $N_1 \approx N(np, np(1-p))$. Therefore, RV $\frac{N_1 - np}{\sqrt{np(1-p)}}$ has approximately standard normal distribution $N(0,1)$.

(b)

$$\left[\frac{n_1 - np}{\sqrt{np(1-p)}}\right]^2 - \left[\frac{(n_1 - np)^2}{np} + \frac{(n - n_1 - n(1-p))^2}{n(1-p)}\right]$$

$$= \frac{n_1^2 - 2nn_1p + n^2p^2 - n_1^2(1-p) + 2nn_1p(1-p) - n^2p^2(1-p) - n^2p}{np(1-p)}$$

$$+\frac{2nn_1p - n_1^2p + 2n^2p(1-p) - 2nn_1p(1-p) - n^2p(1-p)^2}{np(1-p)}$$

$$= \frac{n_1^2p + n^2p^3 + n^2p - 2n^2p^2 - n_1^2p - n^2p + 2n^2p^2 - n^2p^3}{np(1-p)} = 0.$$

Therefore, the distribution of statistic Z coincides with that of $\left(\frac{N_1 - np}{\sqrt{npq}}\right)^2$. The last RV has (approximately) χ^2 distribution with one degree of freedom by the definition:

$$f_{\chi^2}(x) = \frac{1}{\sqrt{2\pi x}}e^{-x/2}, \qquad x > 0.$$