1. (a) A class is made up of 35 students, 20 girls and 15 boys. It is decided to distribute 4 complementary tickets one at a time. It is possible for the same student to receive more than one ticket. What is the probability that (i) all 4 tickets go to girl(s), (5 marks) (ii) 2 tickets go to boy(s) and 2 to girl(s) (5 marks) ?

   (b) In a given population, on the average, one in ten thousand has a rare genetic disease. Using the Poisson approximation find the approximate probability that in a group of two thousand there will be at most three individuals with the disease. (6 marks)

   (c) Explain why it is appropriate to use the Poisson approximation in part b) but not in part a). (4 marks)

2. Profit X (in millions of pounds) on an investment is often modelled as a random variable with density

$$f(x) = A/x^k, \qquad x > B > 0,$$

   where A, B and k are appropriate positive constants. Does $E(X)$ exist if $k = 3$? (1 mark) Does $Var(X)$ exist in this case? (2 marks) Justify your answers.

   Suppose

$$f(x) = 4/x^5, \ x > 1.$$

   (a) Find the expected profit $E(X)$ and its standard deviation. (7 marks)

   (b) Suppose the company must pay 31% in taxes on the profit, so that the net profit is $0.69X$. What is the *expected* net profit and its variance? (5 marks)

   (c) The net profit is invested in a new project which yields £p^2 on an investment of £p. Let Y be the profit (before tax) on this new project. What is the expected value of Y? (5 marks)

3. Let X_1 and X_2 be two random variables with joint density

$$f_{X_1,X_2}(x_1, x_2) = e^{-(x_1+x_2)}, \qquad x_1 > 0,\ x_2 > 0$$

Consider the transformed random variables

$$Y_1 = X_1 - X_2, \quad Y_2 = X_1 + X_2.$$

Find the joint density of Y_1 and Y_2 (16 marks) and draw the region where this density is positive. (4 marks)

4. A device contains two components A and B whose lifetimes (ie periods of time until failure) in *years* are denoted X and Y, respectively. Suppose the random variables X and Y have joint density function

$$f(x, y) = K(x + y)e^{-x}, \qquad x \geq 0,\ 0 \leq y \leq 1.$$

(a) Determine the constant K. (4 marks)

(b) Find the marginal densities of X and Y. (7 marks)

(c) Find the probability that component B (with lifetime Y) fails in the first six months of operation. (3 marks)

(d) Find the conditional density of X given $Y = 1/2$ (3 marks) and compute the mathematical expectation of X given $Y = 1/2$. (3 marks)

5. State the definition of the covariance of two arbitrary random variables X and Y. (2 marks)

Give the formula for $\text{Var}(X+Y)$ in the general case when X and Y are not necessarily independent. (2 marks)

Suppose X and Y are discrete random variables with the joint probability mass function given in the following table:

	$Y=-1$	$Y=1$
$X=-2$	0.4	0.1
$X=2$	0.1	0.4

Find the marginal probability mass functions of X and Y (4 marks) and hence $E(X)$, $E(Y)$, $\text{Var}(X)$ and $\text{Var}(Y)$. (4 marks)

Find $\text{Cov}(X,Y)$ (5 marks) and $\text{Var}(X+Y)$ (3 marks).

6. (a) Define the moment generating function (m.g.f.) of a random variable X. (2 marks)

(b) Suppose $X_1,\ldots,X_n$ are independent and identically distributed uniform RVs on (0,1), so that

$$f_X(x) = 1, \quad 0 \le x \le 1.$$

Obtain the moment generating function, mean and variance of X_i, $i=1,2,\ldots,n$. (6 marks)

(c) Suppose we standardise the sum of the X_i as follows,

$$Y = \frac{\sum_{i=1}^{n} X_i - \frac{n}{2}}{\sqrt{n/12}}.$$

Derive the moment generating function of Y (5 marks) and show that

$$\lim_{n\to\infty} M_Y(t) = e^{\frac{t^2}{2}}.$$

(7 marks)

Hint: Introduce the new varialbe $u = \frac{t\sqrt{3}}{\sqrt{n}}$, replace n with $\frac{3t^2}{u^2}$ and compute the limit $\lim_{u\to 0} M_Y(t)$.

Remark. Note that $e^{t^2/2}$ is the m.g.f. of a standard normal RV. Therefore, a solution to (c) can be viewed as the proof of the simplified version of the Central Limit Theorem!

7. Let X be a Bernoulli random variable with probability of success p. Suppose you have made n independent observations on X: $X_1, X_2, \ldots, X_n$ and N_1 is the (random) number of successes. Suppose also that n is fixed and large enough.

(a) What is the exact distribution of N_1? (3 marks) What is the approximate distribution of $\dfrac{N_1 - np}{\sqrt{np(1-p)}}$? (4 marks)

(b) What is the distribution of the statistic

$$Z = \frac{(N_1 - np)^2}{np} + \frac{(N_0 - n(1-p))^2}{n(1-p)},$$

where $N_0 = n - N_1$? (13 marks)

Hint: Firstly, show that

$$\left[\frac{n_1 - np}{\sqrt{np(1-p)}}\right]^2 - \left[\frac{(n_1 - np)^2}{np} + \frac{(n - n_1 - n(1-p))^2}{n(1-p)}\right] = 0.$$