SECTION A

1. Say what it means for $\{v_1, \ldots, v_k\}$ to *span* a vector space V.

Let U be the subspace of $\mathbf{R}^3$ spanned by $u_1 = (1, 0, -1), u_2 = (1, -2, 1)$ and $u_3 = (2, 2, -4)$. Let W be the set of vectors (x, y, z) in $\mathbf{R}^3$ where $x + y + z = 0$. Show that W is a subspace of $\mathbf{R}^3$. Calculate the dimensions of U and of W. Find the subspace $U \cap W$ and determine its dimension. Determine the subspace $U + W$ and decide whether or not $\mathbf{R}^3 = U \oplus W$. [10 marks]

2. Define the terms: *group, homomorphism, kernel, image.*

Let G be the group of all 3×3 matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$; $a, b, c \in \mathbf{R}$, under the operation of matrix multiplication. Let H be the group of real numbers, under the operation of addition [you need not show that these are groups]. Let $\phi : G \to H$ be defined by

$$\phi\left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\right) = a.$$

Show that ϕ is a homomorphism. Find the kernel and image of ϕ.

[10 marks]

3. Let V be the vector space of polynomials in x of degree at most 3 with coefficients in $\mathbf{R}$. Let the linear map $L : V \to V$ be defined by

$$L(a + bx + cx^2 + dx^3) = d + cx + bx^2 + ax^3$$

Find M, the matrix representing L with respect to the basis $\{1, x, x^2, x^3\}$. What are the eigenvalues and corresponding eigenvectors of M? [10 marks]

4. Let ϕ be the linear map which corresponds to rotation of the plane anti-clockwise through an angle of $90°$ about the origin O. Determine how ϕ maps each of the unit vectors, $(1, 0)$ and (0,1). Hence calculate the matrix M of the linear map ϕ. Let σ_ℓ denote the linear map representing the isometry which is reflection of the plane in the line ℓ with equation $x = 0$, and σ_k correspond to reflection of the plane in the line k with equation $x = y$. Write down the matrices A, B of σ_ℓ, σ_k respectively. Compute the matrix C of the composite map $\sigma_\ell \sigma_k$ (where $\sigma_\ell \sigma_k(x) = \sigma_\ell(\sigma_k(x))$) and decide whether this composite map is itself a reflection or not. Find the smallest positive integer n such that C^n is the identity matrix, and interpret this geometrically. [10 marks]

5. Let f be the bilinear form on $\mathbf{R}^2$ defined by

$$f((x_1,x_2),(y_1,y_2)) = x_1y_1 - x_1y_2 + x_2y_2.$$

Let $u_1 = (2,2), u_2 = (0,1)$. Compute $f(u_1,u_1), f(u_1,u_2), f(u_2,u_1), f(u_2,u_2)$. Find the matrix A of f relative to the basis $\{u_1,u_2\}$. Find the matrix B of f relative to the basis $\{v_1,v_2\}$, where $v_1 = (1,1), v_2 = (0,-1)$.

Find the change of basis matrix P from $\{u_1,u_2\}$ to $\{v_1,v_2\}$ and show that $B = P^TAP$. [9 marks]

6. Define the *rank* and *nullity* of a linear map.

Let $f : \mathbf{R}^3 \to \mathbf{R}^3$ be given by

$$f(x,y,z) = (x+y-z, x-y+2z, 2x+z).$$

Find a basis for the image of f and a basis for the kernel of f. Hence compute the rank of f and the nullity of f. [6 marks]

SECTION B

7. Let V be the vector space of polynomials of degree at most 3. Let U be the subset of V defined by

$$U = \{(a + bx + (a+b)x^2 + dx^3) : a, b, d \in \mathbf{R}\}$$

and W be defined by

$$W = \{a + ax + ax^2 + ax^3 : a \in \mathbf{R}\}.$$

Prove that U and W are subspaces of V. Find the dimensions of each of $U, W, U \cap W$ and $U + W$. Is it true or false that $V = U \oplus W$?

[15 marks]

8. Suppose that $\{x_1, x_2, \ldots, x_n\}$ is a basis for a vector space V. Describe the dual space V^* and describe how to define addition and scalar multiplication on V^* [you need not prove that V^* is a vector space]. Define the dual basis $\{\phi_1, \ldots, \phi_n\}$ to $\{x_1, \ldots, x_n\}$ and prove that it is a basis for V^*.

Consider the basis $\{v_1, v_2, v_3\}$ for $\mathbf{R}^3$ where

$$v_1 = (1, 1, 1), \quad v_2 = (1, 2, 4) \quad \text{and } v_3 = (1, -1, 1).$$

Find the dual basis $\{\phi_1, \phi_2, \phi_3\}$ to $\{v_1, v_2, v_3\}$ and find an expression for the value of each of the three maps at a general point of $\mathbf{R}^3$. Hence compute the values of $\phi_1(3, 2, 1)$, $\phi_2(3, 2, 1)$ and $\phi_3(3, 2, 1)$.

[15 marks]

9. Consider the quadratic form

$$q(x, y, z) = x^2 + 6xy + y^2 + 4z^2.$$

Write down the matrix A representing q with respect to the standard basis. Find a diagonal matrix D equivalent to A and an orthogonal matrix P which describes the change of basis from the standard basis to a basis in which q is diagonal. Describe geometrically the surface $q(x, y, z) = 25$. Draw a sketch of the surface.

[15 marks]

10. (i) Let G be a group. Show that the identity element e is unique.

(ii) Show that, for any $\alpha, \beta, \gamma \in G$, $\alpha * \beta = \alpha * \gamma \Rightarrow \beta = \gamma$. Deduce that no element can be repeated in the same row inside a group table. Similarly show that no element can be repeated in the same column of the table.

(iii) Let X be a set with five elements, $\{e, a, b, c, d\}$, with an operation $\circ$ which satisfies the following table:

$\circ$	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	d	e	a	c
c	c	b	d	e	a
d	d	c	a	b	e

Find an example to show that $\circ$ is not an associative operation.

(iv) Suppose now that G is a set with five elements $\{E, A, B, C, D\}$ with E being an identity element for G, the square of each element being E and the elements labelled so that $A \circ B = C$. By considering the possible multiplication table for G, decide whether or not it is possible for G to be a group. [15 marks]