MATH241 Exam January 2000, Solutions

1.

a) Neither increasing nor decreasing. Bounded above and below. Supremum and maximum are 2, Infimum and minimum are 0.

b) Neither increasing nor decreasing. Bounded below but not above. Infimum and minimum are -2, no supremum or maximum.

2. $a_0 = 3$, $a_1 = 7$, $a_2 = 15$, $a_3 = 1$.

The formulae for the convergents p_n/q_n give $p_0 = 3$, $p_1 = 22$, $p_2 = 333$, and $p_3 = 355$; and $q_0 = 1$, $q_1 = 7$, $q_2 = 106$, and $q_3 = 113$. Thus the first four convergents are $3/1$, $22/7$, $333/106$, and $355/113$.

3. a) Neither; b) Closed; c) Open.

4. f^4 has $3^4 = 81$ fixed points, of which $3^2 = 9$ are also fixed points of f^2, and hence are not period 4 points. Thus f has 72 period 4 points, which constitute $72/4 = 18$ period 4 orbits.

5. The transition matrix is

$$\begin{pmatrix} \frac{2}{3} & \frac{1}{4} \\ \\ \frac{1}{3} & \frac{3}{4} \end{pmatrix},$$

where the first row and column correspond to OK, and the second to Right Then.

The long term proportion of time spent in each of the two states is given by a probability column eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ of this matrix. Thus

$$\begin{aligned} 2x/3 + y/4 &= x \\ x + y &= 1, \end{aligned}$$

with solutions $x = 3/7$ and $y = 4/7$. Thus he starts 3/7 of his lectures with OK in the long run.

6. Fixed points are given by $1 - x^2 = x$, or $x^2 + x - 1 = 0$, with solutions $x = (-1 \pm \sqrt{5})/2$.

Period two points are points which aren't fixed, but are solutions of $f(f(x)) = x$, or $1 - (1 - x^2)^2 = x$, or $x^4 - 2x^2 + x = 0$. It is clear that $x = 0$ and $x = 1$ are solutions of this equation, and are hence period 2 points: the other two solutions must be the fixed points of f.

7. $f(t) - 1/2$ is an odd function, so the Fourier series expansion of $f(t)$ is of the form

$$\frac{1}{2} + \sum_{r=1}^{\infty} b_r \sin rt,$$

where

$$\begin{aligned} b_r &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin rt\, dt \\ &= \frac{1}{\pi}\int_{0}^{\pi} \sin rt\, dt \\ &= \frac{1}{\pi}\left[\frac{\cos rt}{r}\right]_{\pi}^{0} \\ &= \frac{1-(-1)^r}{\pi r}. \end{aligned}$$

Hence the Fourier series expansion of $f(t)$ is

$$\frac{1}{2} + \sum_{r=1}^{\infty} \frac{1-(-1)^r}{\pi r} \sin rt.$$

8. The coefficients of the Fourier expansion of $|t|$ are $a_0 = \pi$, $a_r = 2((-1)^r - 1)/r^2\pi$ for $r \geq 1$, and $b_r = 0$ for all r. Parseval's theorem states that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)^2\, dt = \frac{a_0^2}{4} + \frac{1}{2}\sum_{r=1}^{\infty}(a_r^2 + b_r^2).$$

In this case, we have

$$\int_{-\pi}^{\pi} f(t)^2\, dt = \int_{-\pi}^{\pi} t^2\, dt = 2\frac{\pi^3}{3},$$

so Parseval's theorem gives

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{1}{2\pi^2}\sum_{r \text{ odd}} \frac{16}{r^4},$$

Or

$$\frac{\pi^4}{96} = \sum_{r \text{ odd}} \frac{1}{r^4}.$$

9. We use the following theorem from lectures: Suppose that $f\colon [a,b] \to \mathbf{R}$ is an increasing function, that a sequence (x_n) is defined iteratively using f from some starting value $x_0 \in [a,b]$, and that $x_1 \geq x_0$. If there is a fixed point of f in $[x_0, b]$, then (x_n) is an increasing sequence which tends to a limit l, which is the smallest fixed point of f in $[x_0, b]$.

In this example $f(x) = x + 1 - x^2/5$, so the fixed points of f are $\pm\sqrt{5}$. Also $f'(x) = 1 - 2x/5$, which is non-negative for $x \leq 5/2$, so we work in the interval $[a,b] = [2, 5/2]$, which contains $\sqrt{5}$. . Since $x_1 = 2+1-4/5 > x_0$, it follows from the theorem that (x_n) is increasing and tends to $\sqrt{5}$ as $n \to \infty$.

The completeness axiom states that any non-empty subset of $\mathbf{R}$ which is bounded above has a least upper bound (and any non-empty subset of $\mathbf{R}$ which is bounded below has a greatest lower bound).

Let (x_n) be an increasing sequence which is bounded above: by the completeness axiom, it has a least upper bound l (so $x_n \leq l$ for all n). Given any $\epsilon > 0$, there is some N such that $l - \epsilon < x_N \leq l$ (otherwise $l - \epsilon$ would be an upper bound), and since (x_n) is increasing this means that $l - \epsilon < x_n \leq l$ for all $n \geq N$. Thus $x_n \to l$ as $n \to \infty$.

10. S is countable if there is a one-to-one correspondence between it and $\mathbf{N}$: or equivalently if there is a sequence (x_n) which includes all of the elements of S.

To show that $\mathbf{Q}$ is countable, define a sequence (y_n) by $y_{j(j+1)/2+k} = (1+k)/(1+j-k)$ for $j \geq 0$ and $0 \leq k \leq j$. This sequence includes all positive rationals, since if $p, q > 0$ then $y_n = p/q$ when $n = (p^2+2pq+q^2-p-3q-2)/2$. Hence the sequence $(x_n) = (0, y_0, -y_0, y_1, -y_1, \ldots)$ includes every element of $\mathbf{Q}$.

To show that $\mathbf{R}$ is uncountable, it is enough to show that the set S of real numbers in $[0,1]$ whose decimal expansions only involve the digits 5 or 6 is uncountable. Let (x_n) $(n \geq 1)$ be any sequence of elements of S, and let the ith decimal digit of x_n be d_n^i. Then

$$x = \sum_{i=1}^{\infty} \frac{11 - d_i^i}{10^i}$$

is an element of S which is not included in the sequence. Hence S is uncountable.

To show that the set S in part c) is uncountable, let (x_n) be any sequence of subsets of $\mathbf{N}$. Let $A = \{n \in \mathbf{N} : n \notin x_n\}$. Then $A \in S$ is not included in the sequence (x_n).

11. Let $\triangleright$ be the order on the positive integers given by

$$3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright 11 \triangleright \dots$$

$$6 \triangleright 10 \triangleright 14 \triangleright 18 \triangleright 22 \triangleright \dots$$

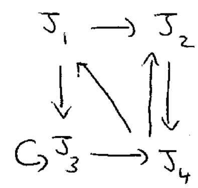
$$12 \triangleright 20 \triangleright 28 \triangleright 36 \triangleright 44 \triangleright \dots$$

$$\dots$$

$$\dots \triangleright 16 \triangleright 8 \triangleright 4 \triangleright 2 \triangleright 1.$$

If $f: [0,1] \to [0,1]$ is continuous and has a periodic orbit of period n, then it also has periodic orbits of period m for all m with $n \triangleright m$.

The Markov graph of $(1\,2\,4\,3\,5)$ is



There is a loop $J_1 \to J_3 \to J_4 \to J_1$ of length 3, and hence there is a fixed point x of f^3 with itinerary $k(x) = (134)^\infty$. Since this is not the itinerary of a fixed point, f has a period 3 orbit, and hence by Sharkovsky's theorem has periodic orbits of every period.

If $\pi \in S_5$ is a pattern of a periodic orbit of a unimodal map, then it is a cyclic permutation which is unimodal: i.e. it is increasing on $\{1, 2, \dots k\}$ and decreasing on $\{k, k+1, \dots, 5\}$ for some $k \in \{1, 2, 3, 4, 5\}$.

Let $\pi \in S_5$ be a unimodal permutation.

If $k = 1$ or $k = 5$ then π cannot be cyclic (in the former case $\pi(1) = 5$ and $\pi(5) = 1$, and in the latter π is the identity).

Clearly either $\pi(1) = 1$ or $\pi(5) = 1$, and hence if π is cyclic then $\pi(5) = 1$.

If $k = 2$ then $\pi(2) = 5$ and π is determined by $\pi(1)$: inspecting the three possibilities, only $(1\,3\,4\,2\,5)$ is cyclic.

If $k = 3$ then $\pi(3) = 5$ and π is determined by $\pi(4)$: inspecting the three possibilities, only $(1\,2\,4\,3\,5)$ is cyclic.

If $k = 4$ then $\pi(4) = 5$ and hence $\pi = (1\,2\,3\,4\,5)$.

Thus the three period 5 patterns which can arise from a unimodal map are $(1\,3\,4\,2\,5)$, $(1\,2\,4\,3\,5)$, and $(1\,2\,3\,4\,5)$.

12.

a) $f: \mathbf{R} \to \mathbf{R}$ is even if $f(t) = f(-t)$ for all $t \in \mathbf{R}$.

If $f(t)$ is even then $f(t)\sin rt$ is odd for all r, and hence the coefficients

$$b_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin rt\,dt$$

of the sine terms in its Fourier series expansion vanish for all r.
b) $|\sin t|$ is an even function, so its Fourier series expansion is of the form

$$\frac{a_0}{2} + \sum_{r=1}^{\infty} a_r \cos rt,$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin t|\,dt = \frac{2}{\pi}\int_0^{\pi} \sin t\,dt = 4/\pi,$$

and

$$a_r = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin t|\cos rt\,dt = \frac{2}{\pi}\int_0^{\pi} \sin t\cos rt\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi} (\sin(r+1)t - \sin(r-1)t)\,dt = \frac{1}{\pi}(1-(-1)^{r+1})\frac{-2}{r^2-1}$$

for $r > 1$, and

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin t\cos t\,dt = 0.$$

Hence the Fourier series expansion is

$$\frac{2}{\pi} - \frac{2}{\pi}\sum_{r=2}^{\infty} \frac{1-(-1)^{r+1}}{r^2-1}\cos rt.$$

c) The series converges pointwise if for all $t^* \in \mathbf{R}$ and all $\epsilon > 0$, there is an N such that

$$\left| f(t^*) - \sum_{r=1}^{n} f_r(t^*) \right| < \epsilon$$

for all $n \geq N$.

It converges uniformly if for all $\epsilon > 0$ there is an N such that

$$\left| f(t) - \sum_{r=1}^{n} f_r(t) \right| < \epsilon$$

for all $n \geq N$ and all $t \in \mathbf{R}$.

The Fourier Series Theorem says that if $f(t)$ is continuous and piecewise differentiable, then its Fourier series expansion converges to it uniformly.

Since $|\sin t|$ is continuous and piecewise differentiable, the convergence in this case is uniform.

13. The coefficients c_r are given by

$$\begin{aligned} c_r &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-irt}\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-ir)t}\,dt \\ &= \frac{1}{2\pi(1-ir)}(e^{\pi}(-1)^r - e^{-\pi}(-1)^r) \\ &= \frac{(-1)^r \sinh\pi}{\pi(1-ir)}. \end{aligned}$$

Hence the Fourier series expansion of $f(t)$ is

$$\sum_{r=-\infty}^{\infty} \frac{(-1)^r \sinh\pi}{\pi(1-ir)} e^{irt}.$$

To apply Parseval's theorem, observe that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)^2\,dt = \frac{1}{4\pi}(e^{2\pi} - e^{-2\pi}) = \frac{1}{2\pi}\sinh(2\pi),$$

and

$$|c_r|^2 = \left(\frac{(-1)^r \sinh\pi}{\pi(1+r^2)}\right)^2 (1+r^2) = \frac{\sinh^2\pi}{\pi^2(1+r^2)}.$$

Hence

$$\sum_{r=-\infty}^{\infty} \frac{\sinh^2\pi}{\pi^2(1+r^2)} = \frac{1}{\pi}\sinh\pi\cosh\pi,$$

or

$$\sum_{r=-\infty}^{\infty} \frac{1}{1+r^2} = \pi\coth\pi$$

as required.