MATH227 MATHEMATICAL METHODS FOR NON-PHYSICAL SYSTEMS

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1.

$$U(x,y) = xy + 7x + 3y = U_0 \Rightarrow y = \frac{U_0 - 7x}{x+3} = -7 + \frac{U_0 + 21}{x+3}.$$

$$\frac{dy}{dx} = -\frac{U_0+21}{(x+3)^2} < 0,$$

$$\frac{d^2y}{dx^2} = 2\frac{U_0+21}{(x+3)^3} > 0,$$

2. Budget constraint touches indifference curve where

$$\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}} = \frac{2}{3}$$

$$\frac{3(x+3)^2(y+2)^5}{5(x+3)^3(y+2)^4} = \frac{2}{3}$$

$$9(y+2) = 10(x+3) \Rightarrow 10x - 9y = -12.$$

Solving with $2x + 3y = 36$, we have $x = 6$, $y = 8$.

3. $c(x,y)$ is minimised where

$$\frac{\frac{\partial q}{\partial x}}{\frac{\partial q}{\partial y}} = \frac{\frac{\partial c}{\partial x}}{\frac{\partial c}{\partial y}}$$

$$\frac{\partial q}{\partial x} = \frac{(y+4)(x+y+9) - (x+5)(y+4)}{(x+y+9)^2}$$

$$= \frac{(y+4)^2}{(x+y+9)^2}$$

$$\frac{\partial q}{\partial y} = \frac{(x+5)^2}{(x+y+9)^2}$$

$$\Rightarrow \frac{(y+4)^2}{(x+5)^2} = \tfrac{16}{9} \Rightarrow \frac{y+4}{x+5} = \tfrac{4}{3}.$$

$$\Rightarrow 3 = \frac{\frac{4}{3}}{1+\frac{4}{3}}(x+5) = \tfrac{4}{7}(x+5)$$

$$\Rightarrow x = \tfrac{1}{4}, \quad y = 3$$

$$\Rightarrow c = 16\tfrac{1}{4} + 9.3 + 4 = 35.$$

Minimum cost for production level of 3 units is 35 units.

4.

$$C(q) = q^3 - 6q^2 + 14q + 10.$$

(i) Fixed cost $C(0) = 10$. (ii) $MC(q) = C'(q) = 3q^2 - 12q + 14$.

(iii) $AVC(q) = \frac{C(q)-C(0)}{q} = q^2 - 6q + 14$.

Cease production when $p = \min(AVC)$.

$$AVC'(q) = 2q - 6 = 0 \quad \text{when} \quad q = 3 \Rightarrow p = AVC(3) = 5.$$

5.

$$S(p) = 4\frac{5p+2}{5p+8} \Rightarrow \frac{dS}{dp} = 4\frac{5(5p+8) - 5(5p+2)}{(5p+8)^2} = \frac{120}{(5p+8)^2} > 0.$$

For a tax-rate of t, equilibrium is where

$$S((1-t)p) = D(p) \Rightarrow 4\frac{5.\frac{4}{5}p + 2}{5\frac{4}{5}p + 8} = \sqrt{21 - 3p}$$

$$\Rightarrow 2\frac{2p+1}{p+2} = \sqrt{21 - 3p}.$$

$p = 4$ is a solution by inspection. Since $S(p)$ is increasing, $D(p)$ is decreasing, it is the only solution. Then amount sold in a week $= D(4) = 3$.

6.

$$C(q) = q^3 - 8q^2 + 24q + 3, \quad D(p) = 48 - p = q \Rightarrow p = 48 - q.$$

Profit given by

$$P(q) = pq - C(q) = (48 - q)q - (q^3 - 8q^2 + 24q + 3) = -q^3 + 7q^2 + 24q - 3$$

$$\Rightarrow P'(q) = -3q^2 + 14q + 24 = -(3q + 4)(q - 6) = 0 \quad \text{for} \quad q = -\tfrac{4}{3}, \quad 6.$$

Take the +ve solution $q = 6$. Then $p = 48 - q = 42$.

$$P''(q) = -6q + 14 < 0 \quad \text{for} \quad q = 6.$$

So we have a local maximum. Also

$$P(6) = -6^3 + 7.6^2 + 24.6 - 3 = 177 > P(0) = -3.$$

So $q = 6$ is a global maximum.

7. Have

$$\begin{aligned} y =& p(1-x) \Rightarrow px + y = p \\ \Rightarrow \frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}} =& p \\ \Rightarrow \frac{y+9}{x+2} =& p \Rightarrow p(x+2) = y+9 \\ \Rightarrow x =& \frac{9}{2p} - \tfrac{1}{2} = \frac{9-p}{2p}. \end{aligned}$$

$$\Rightarrow \frac{dx}{dp} = -\frac{9}{2p^2}.$$

so x decreases with p for $p < 9$.

8.

$$\frac{dn}{dt} = -21n + 10n^2 - n^3 = -n(n-3)(n-7) = f(n).$$

Equilibrium densities $n = 0$, $n = 3$, $n = 7$.

$f'(0) < 0$, $f(n) \to -\infty$ as $n \to \infty$. So graph looks like this:

Equilibria at $n = 0$, $n = 7$ stable; equilibrium at $n = 3$ unstable.

9.

$$\frac{dx}{dt} = x(4 - 2x + 3y), \quad \frac{dy}{dt} = y(3 + x - 4y),$$

$$\begin{aligned} 4 - 2x + 3y = 3 + x - 4y =& 0 \Rightarrow x = 5, \quad y = 2, \\ \text{or} \quad y = 4 - 2x + 3y =& 0 \Rightarrow x = 2, \\ \text{or} \quad x = 3 + x - 4y =& 0 \Rightarrow y = \tfrac{3}{4}, \\ \text{or} \quad x = y =& 0. \end{aligned}$$

So the equilibria are $(5, 2)$, $(2, 0)$, $(0, \frac{3}{4})$, $(0, 0)$.

10.

$$\mathbf{x} = \mathbf{x}^e + c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t},$$

where $\lambda_{1,2}$ are the e-values, $\mathbf{x}_{1,2}$ are the e-vectors. If λ_1 and λ_2 are complex, then we have an spiral point; stable if the real parts of the e-values are $-$ve, unstable if they are $+$ve.

11. Budget constraint touches indifference curve where

$$\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}} = \frac{p}{q}$$

$$\frac{\frac{1}{3}(x+9)^{-\frac{2}{3}}(y+5)^{\frac{2}{3}}}{\frac{2}{3}(x+9)^{\frac{1}{3}}(y+5)^{-\frac{2}{3}}} = \frac{p}{q} \Rightarrow \frac{1}{2}\frac{y+5}{x+9} = \frac{p}{q}$$

$$\Rightarrow 2p(x+9) = q(y+5)$$

Adding to budget constraint $px + qy = 15$ we have

$$3px = 5q + 15 - 18p \Rightarrow x = \frac{15+5q-18p}{3p} = \frac{15+5q}{3p} - 6$$

$$\Rightarrow y = \frac{30+18p-5q}{3q} = \frac{10+6p}{q} - \frac{5}{3}.$$

$$\epsilon_x = p\frac{3p}{15-18p+5q}\left(-\frac{15+5q}{3p^2}\right) = \frac{15+5q}{18p-5q-15}$$

$$\Rightarrow \epsilon_x + 1 = \frac{18p}{18p-15-5q} < 0 \quad \text{if} \quad 18p - 5q < 15.$$

12.

$$C(q) = q^3 - 2q^2 + 5q + 36 \Rightarrow AVC(q) = q^2 - 2q + 5.$$
$$AVC'(q) = 2q - 2 = 0 \quad \text{when} \quad q = 1 \Rightarrow \min(AVC) = 4.$$

So cease production when $p = \min(AVC) = 4$. For $p \geq 4$,

$$p = C'(q) = 3q^2 - 4q + 5 \Rightarrow 3q^2 - 4q + 5 - p = 0$$
$$\Rightarrow q = \frac{4 \pm \sqrt{16 - 12(5-p)}}{6} = \frac{2 \pm \sqrt{3p - 11}}{3}.$$

Take +ve sign for maximum profit. So

$$S(p) = \begin{cases} \frac{2+\sqrt{3p-11}}{3} & \text{if } p \geq 4 \\ 0 & \text{if } p < 4. \end{cases}$$

Equilibrium is when $NS(p) = D(p)$ (N firms) so

$$\frac{2 + \sqrt{3p - 11}}{3} = 11 - p.$$

$p = 9$ is a solution by inspection, and since $D(p)$ is decreasing and $S(p)$ is increasing, it is unique.

$$p = 9 \Rightarrow q = \frac{1}{N}D(p) = 2 \Rightarrow P(q) = pq - C(q)$$
$$= 18 - (8 - 8 + 10 + 36) = -28.$$

So each firm makes a loss of 28 units.

Production not viable in the long-run for $p < \min(ATC)$.

$$ATC = q^2 - 2q + 5 + \frac{36}{q} \Rightarrow ATC'(q) = 2q - 2 - \frac{36}{q^2}$$
$$= 0 \quad \text{when} \quad q = 3,$$

by inspection. It is a minimum, since

$$ATC''(q) = 2 + \frac{72}{q^3} > 0.$$
$$\min(ATC) = 9 - 6 + 5 + 12 = 20.$$

So minimum price in the long-run is 20 units.

13.

$$\begin{aligned} C_1(q_1) =& 7 + 6q_1 + q_1^2, \\ C_2(q_2) =& 5 + 9q_2 + q_2^2, \end{aligned}$$

Profits:

$$\begin{aligned} P_1(q_1, q_2) =& pq_1 - (7 + 6q_1 + q_1^2) = [15 - (q_1 + q_2)]q_1 - (7 + 6q_1 + q_1^2) \\ =& -2q_1^2 - q_1 q_2 + 9q_1 - 7, \\ P_2(q_1, q_2) =& pq_2 - (5 + 9q_2 + q_2^2) = [15 - (q_1 + q_2)]q_2 - (5 + 9q_2 + q_2^2) \\ =& -q_1 q_2 - 2q_2^2 + 6q_2 - 5. \end{aligned}$$

Cournot duopoly $\Rightarrow$ maximise P_1, P_2 wrto q_1, q_2 respectively. So

$$\begin{aligned} \frac{\partial P_1}{\partial q_1} =& -4q_1 - q_2 + 9 = 0, \\ \frac{\partial P_2}{\partial q_2} =& -q_1 - 4q_2 + 6 = 0, \end{aligned}$$

Then $q_1 = 2$, $q_2 = 1$. So $p = 15 - 2 - 1 = 12$ and $P_1(2, 1) = 1$, $P_2(2, 1) = -3$.

If co-operate, maximise

$$\begin{aligned} P(q_1, q_2) =& P_1(q_1, q_2) + P_2(q_1, q_2) \\ =& -2q_1^2 - 2q_1 q_2 + 9q_1 - 2q_2^2 + 6q_2 - 12 \\ \frac{\partial P}{\partial q_1} =& -4q_1 - 2q_2 + 9 = 0, \\ \frac{\partial P}{\partial q_2} =& -2q_1 - 4q_2 + 6 = 0, \end{aligned}$$

giving $q_1 = 2$, $q_2 = \frac{1}{2}$. Then $P_1(2, \frac{1}{2}) = 2$, $P_2(3, 1) = -\frac{7}{2}$.

14. Have

$$\int_{N_0}^{N_1^{(j)}} \frac{dn_1}{n_1 - N} = \int_0^{jT} \frac{dt}{T}$$

$$\Rightarrow [\ln(n_1 - N)]_{N_0}^{N_1^{(j)}} = j$$

$$\Rightarrow \frac{N_1^{(j)} - N}{N_0 - N} = e^j$$

$$\Rightarrow N_1^{(j)} = N + (N_0 - N)e^j$$

For the second population, we have

$$\int_{N_2^{(j-1)}}^{2N_2^{(j)}} \frac{dn_2}{n_2} = \int_{(j-1)T}^{jT} \frac{dt}{T}$$

$$\Rightarrow [\ln n_2]_{N_2^{(j-1)}}^{2N_2^{(j)}} = 1$$

$$\Rightarrow N_2^{(j)} = \frac{e}{2} N_2^{(j-1)}$$

$$\Rightarrow N_2^{(j)} = \left(\frac{e}{2}\right)^j N_0$$

.

Since $N_0 > N$ and $e > 2$ it is clear that both populations increase indefinitely. But if the second harvesting strategy is changed so that the population density is reduced by $\frac{2}{3}$ at each step, then

$$N_2^{(j)} = \left(\frac{e}{3}\right)^j N_0$$

and the population dies away since $e < 3$.

15.

$$\frac{dx}{dt} = x(7-2x) - 3xy, \quad \frac{dy}{dt} = y(10-2y) - 4xy,$$

Terms (1), (3) are logistic growth functions, implying each population could survive on its own in a limited resource environment.

Terms (2), (4) with negative signs imply the two species are competing.

$$\begin{aligned} \text{Either} \quad 7-2x-3y =& 10-4x-2y = 0 \Rightarrow x = 2, \quad y = 1 \\ \text{or} \quad x = 10-4x-2y =& 0 \Rightarrow y = 5, \\ \text{or} \quad y = 7-2x-3y =& 0 \Rightarrow x = \tfrac{7}{2}, \\ \text{or} \quad x = y = 0.& \end{aligned}$$

So the equilibria are $(0,0)$, $(0,5)$, $(\frac{7}{2},0)$, $(2,1)$.

Community matrix

$$A = \begin{pmatrix} (7-2x-3y)-2x & -3x \\ -4y & (10-4x-2y)-2y \end{pmatrix}.$$

For $(0,0)$, $A = \begin{pmatrix} 7 & 0 \\ 0 & 10 \end{pmatrix}$. E-values 7, 10 both positive $\Rightarrow$ improper node, unstable.

For $(0,5)$, $A = \begin{pmatrix} -8 & 0 \\ -20 & -10 \end{pmatrix}$. E-values -8, -10 both negative $\Rightarrow$ improper node, stable.

For $(\frac{7}{2},0)$, $A = \begin{pmatrix} -7 & -\frac{21}{2} \\ 0 & -4 \end{pmatrix}$. E-values -7, -4 both negative $\Rightarrow$ improper node, stable.

For $(2,1)$, $A = \begin{pmatrix} -4 & -6 \\ -4 & -2 \end{pmatrix}$.

Linearised equations

$$\begin{aligned} \frac{d\epsilon_x}{dt} =& -4\epsilon_x - 6\epsilon_y \\ \frac{d\epsilon_y}{dt} =& -4\epsilon_x - 2\epsilon_y. \\ -4\epsilon_x - 6\epsilon_y =& -4\delta[3e^{-8t} - 2e^{2t}] - 12\delta[e^{-8t} + e^{2t}] \\ =& \delta[-24e^{-8t} - 4e^{2t}] = \frac{d\epsilon_x}{dt} \\ -4\epsilon_x - 2\epsilon_y =& -4\delta[3e^{-8t} - 2e^{2t}] - 4\delta[e^{-8t} + e^{2t}] \\ =& \delta[-16e^{-8t} + 4e^{2t}] = \frac{d\epsilon_x}{dt} \end{aligned}$$

Also $\epsilon_x(0) = \delta$, $\epsilon_y(0) = 4\delta$.