MATH227 MATHEMATICAL METHODS FOR NON-PHYSICAL SYSTEMS

JANUARY 2005

Candidates should attempt the whole of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

SECTION A

1. The utility function U is defined by

$$U(x,y) = xy + 3x + 5y, \quad x \geq 0, y \geq 0.$$

Express the indifference curve $U(x,y) = U_0$ in the explicit form

$$y = f(x, U_0),$$

for some function f.

Verify that

$$\frac{df}{dx} < 0 \quad \text{and} \quad \frac{d^2 f}{dx^2} > 0,$$

within the domain of U.

Sketch the indifference curve $U_0 = 15$.

[6 marks]

2. A consumer has utility function

$$U(x,y) = (x+1)^2(y+3)^3$$

and is subject to the budget constraint

$$x + 3y = 10$$

where x and y denote the amounts of two commodities. Determine the amounts purchased.

Show that the budget constraint line touches the indifference curve $U = 4096$.

[6 marks]

3. A firm has production function

$$q(x,y) = (x+1)^{\frac{4}{5}} y^{\frac{1}{5}}$$

and cost function

$$c(x,y) = 4x + 3y + 2$$

per unit time, with inputs x and y. Find the minimum value of $c(x,y)$ consistent with a fixed production level of 3 units per unit time.

[6 marks]

4. The total cost function for a single-commodity firm is

$$C(q) = q^3 - 6q^2 + 14q + 10,$$

where q is the quantity of the commodity produced in unit time.

Determine:

(i) The fixed cost;

(ii) The marginal cost function, $MC(q)$;

(iii) The average variable cost function, $AVC(q)$.

Find the price at which the firm would be forced to cease production in the *short-run* in an ideal competitive market.

[6 marks]

5. In an ideal competitive market for one good, the weekly supply function is given by

$$S(p) = 8\frac{p+2}{p+4}$$

where p is the price per unit of the good, and the demand function is

$$D(p) = \sqrt{48 - 3p},$$

for $p < 16$, where p is the price per unit good.

Verify that $S(p)$ is an increasing function of p.

Determine the equilibrium price, and the amount sold in a week.

[6 marks]

6. A monopolist has cost function

$$C(q) = q^3 - 5q^2 + 9q + 3,$$

for the production of a quantity q of a commodity, per unit time. Write down the profit function, as a function of q, when the demand function is

$$D(p) = 12 - p, \quad p < 12.$$

Determine the market price. Verify that your answer maximises the monopolist's profit function for $q \geq 0$.

[8 marks]

7. The population density, $n(t)$, of a species of fish satisfies

$$\frac{dn}{dt} = -14n + 9n^2 - n^3.$$

Find the equilibrium densities, and determine their stability.

[6 marks]

8. A mathematical model of the behaviour of two interacting species X and Y is described by the coupled differential equations

$$\frac{dx}{dt} = x(7 - 3x + y), \quad \frac{dy}{dt} = y(7 - x - 2y),$$

where $x(t)$ and $y(t)$ are the population densities of X and Y respectively. Find the equilibria–you are not required to classify them.

[6 marks]

9. The community matrix at a particular equilibrium point of a two-species model has distinct eigenvalues with opposite signs. Describe briefly, with the aid of diagrams, the possible local behaviour of trajectories near to the equilibrium point.

[5 marks]

SECTION B

10. Sam's preferences for crisps and chocolate are quantified by the utility function

$$U(x,y) = (x+4)(y+1) - 2,$$

where x and y denote the numbers of packets of crisps and bars of chocolate eaten per week, respectively.

Show that Sam prefers N bars of chocolate, with no crisps, to N bags of crisps, with no chocolate.

Express the indifference curve $U(x,y) = U_0$ in the explicit form

$$y = f(x, U_0),$$

for some function f, and sketch a selection of indifference curves for various values of U_0.

Crisps cost £p per bag and chocolate costs £1 per bar. Sam has £7 per week to spend on these snacks. How many bags of crisps and bars of chocolate does Sam buy per week, as a function of p?

The own-elasticity of Sam's demand for crisps, ϵ_x, is defined by

$$\epsilon_x = \frac{p}{x}\frac{dx}{dp}.$$

Compute ϵ_x and show that $\epsilon_x < -1$ if $p < 2$.

[15 marks]

11. Each of N identical firms producing a single commodity has cost function

$$C(q) = q^3 - 2q^2 + 4q + 36.$$

Show that each firm has supply function

$$S(p) = \begin{cases} \frac{2+\sqrt{3p-8}}{3} & \text{if } p \geq 3 \\ 0 & \text{if } p < 3. \end{cases}$$

Find the equilibrium price and the profit/loss made by each firm in an ideal competitive market where the demand function is

$$D(p) = N(10 - p).$$

Find the price below which production is not viable in the *long-run*.

[15 marks]

12. Two firms producing the same commodity have cost functions

$$\begin{aligned} C_1(q_1) &= 6 + 7q_1 + \frac{1}{2}q_1^2, \\ C_2(q_2) &= 4 + 8q_2 + q_2^2, \end{aligned}$$

where q_1 and q_2 are the respective quantities of production per unit time.

The two firms form a Cournot duopoly to supply a market which has demand function $D(p) = 14 - p$. Find the production of each firm. Find the price charged per unit of the good and the profits made by the companies.

The firms tentatively agree to co-operate and maximise their joint profit. Find what the individual profits would be, if such an agreement were carried out.

[15 marks]

13. The population density of a particular species satisfies the growth equation

$$\frac{dn}{dt} = 2n^2 - 7n + 3.$$

Find the equilibrium densities, and determine their stability.

Express the function

$$\frac{1}{2n^2 - 7n + 3}$$

in partial fractions, and hence obtain the solution of the growth equation with initial condition $n(0) = 0$. Describe what happens to the population density as $t \to \infty$.

[15 marks]

14. The population densities of two interacting species evolve with time according to

$$\frac{dx}{dt} = x(5-x) - xy, \quad \frac{dy}{dt} = y(7-2y) - xy,$$

where $x(t)$ and $y(t)$ are the population densities of the two species.

Comment on the biological significance of each of the terms on the right hand sides of these equations.

Show that there are equilibria at $(0,0)$, $(0,\frac{7}{2})$, $(5,0)$, $(3,2)$ and classify the first three of them.

Write down the linearised growth equations in the neighbourhood of the coexistence equilibrium, $(3,2)$, using the substitutions $x = 3 + \epsilon_x$, $y = 2 + \epsilon_y$ where the deviations from equilibrium are small. Verify that the particular solutions corresponding to the initial conditions $x(0) = 3 + \delta$, $y(0) = 2$ are given approximately by

$$x(t) = 3 + \tfrac{1}{5}\delta\left[3e^{-t} + 2e^{-6t}\right], \quad y(t) = 2 + \tfrac{1}{5}\delta\left[-2e^{-t} + 2e^{-6t}\right].$$

[15 marks]