MATH227 MATHEMATICAL METHODS FOR NON-PHYSICAL SYSTEMS

JANUARY 2004

Candidates should attempt the whole of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

SECTION A

1. The utility function U is defined by

$$U(x,y) = xy + 4x + 3y, \quad x \geq 0, y \geq 0.$$

Express the indifference curve $U(x,y) = U_0$ in the explicit form

$$y = f(x, U_0),$$

for some function f.

Verify that

$$\frac{df}{dx} < 0 \quad \text{and} \quad \frac{d^2 f}{dx^2} > 0,$$

within the domain of U.

Sketch the indifference curve $U_0 = 12$.

[6 marks]

2. A consumer has utility function

$$U(x,y) = (x+2)^3(y+1)^2$$

and is subject to the budget constraint

$$2x + y = 5$$

where x and y denote the amounts of two commodities. Determine the amounts purchased.

Show that the budget constraint line touches the indifference curve $U = 432$.

[6 marks]

3. A firm has production function

$$q(x,y) = x^{\frac{1}{4}} y^{\frac{3}{4}}$$

and cost function

$$c(x,y) = 3x + 4y + 10$$

per unit time, with inputs x and y. Find the minimum value of $c(x,y)$ consistent with a fixed production level of 3 units per unit time.

[6 marks]

4. The total cost function for a single-commodity firm is

$$C(q) = q^3 - 8q^2 + 20q + 8,$$

where q is the quantity of the commodity produced in unit time.

Determine:

(i) The fixed cost;

(ii) The marginal cost function, $MC(q)$;

(iii) The average variable cost function, $AVC(q)$.

Find the price at which the firm would be forced to cease production in the *short-run* in an ideal competitive market.

[6 marks]

5. In an ideal competitive market for one good, the weekly supply function is given by

$$S(p) = p + \frac{6p}{p+2}$$

where p is the price per unit of the good, and the demand function is

$$D(p) = \sqrt{72 - 2p},$$

for $p < 36$, where p is the price per unit good.

Verify that $S(p)$ is an increasing function of p.

Determine the equilibrium price, and the amount sold in a week.

[6 marks]

6. A monopolist has cost function

$$C(q) = q^3 - 3q^2 + 5q + 4,$$

for the production of a quantity q of a commodity, per unit time. Write down the profit function, as a function of q, when the demand function is

$$D(p) = 20 - p, \quad p < 20.$$

Determine the market price. Verify that your answer maximises the monopolist's profit function for $q \geq 0$.

[7 marks]

7. The population density, $n(t)$, of a species of fish satisfies

$$\frac{dn}{dt} = -15n + 8n^2 - n^3.$$

Find the equilibrium densities, and determine their stability.

[6 marks]

8. A mathematical model of the behaviour of two interacting species X and Y is described by the coupled differential equations

$$\frac{dx}{dt} = x(5 - 4x + y), \quad \frac{dy}{dt} = y(8 - 2y - x),$$

where $x(t)$ and $y(t)$ are the population densities of X and Y respectively. Find the equilibria–you are not required to classify them.

[6 marks]

9. The community matrix at a particular equilibrium point of a two-species model has distinct eigenvalues of the same sign. Describe briefly, with the aid of diagrams, the possible local behaviour of trajectories near to the equilibrium point.

[6 marks]

SECTION B

10. Bev's preferences for tea and coffee are quantified by the utility function

$$U(x, y) = (xy^2 + 1)^2,$$

where x and y denote the numbers of cups of tea and coffee drunk per day, respectively.

Sketch a selection of indifference curves for U, and verify that the conditions for U to be a utility function are satisfied.

Bev is on holiday in France, where tea costs 3 euros per cup and coffee costs 4 euros per cup. Bev has 18 euros per day to spend on drinks. How many cups of each does Bev drink per day?

Bev now travels to Germany where the price of tea is the same but coffee now costs 6 euros per cup. How do her drinking habits change?

On her last day in Germany she treats herself to a large cream cake costing 9 euros which she pays for out of her drinks budget. How many cups of coffee and tea does she drink that day?

[15 marks]

11. Each of N identical firms producing a single commodity has cost function

$$C(q) = q^3 - 4q^2 + 8q + 18.$$

Show that each firm has supply function

$$S(p) = \begin{cases} \frac{4+\sqrt{3p-8}}{3} & \text{if } p \geq 4 \\ 0 & \text{if } p < 4. \end{cases}$$

Find the equilibrium price and the profit/loss made by each firm in an ideal competitive market where the demand function is

$$D(p) = N(4 - \frac{1}{2}p).$$

Find the price below which production is not viable in the *long-run*.

[15 marks]

12. Two firms producing the same commodity have cost functions

$$\begin{aligned} C_1(q_1) =& 8 + 5q_1 + q_1^2, \\ C_2(q_2) =& 6 + 4q_2 + \frac{3}{2}q_2^2, \end{aligned}$$

where q_1 and q_2 are the respective quantities of production per unit time.

The two firms form a Cournot duopoly to supply a market which has demand function $D(p) = 10 - p$. Find the production of each firm. Find the price charged per unit of the good and the profits made by the companies.

The firms tentatively agree to co-operate and maximise their joint profit. Find what the individual profits would be, if such an agreement were carried out.

However, the second firm decides to trade elsewhere, leaving the first as a monopoly. What is the price now?

[15 marks]

13. The population density of gazelles, n, in a well-defined region of a wildlife reserve satisfies

$$\frac{dn}{dt} = 15n^2 - 5n^3.$$

Find the non-zero equilibrium density, and determine its stability.

A pride of lions starts preying upon the gazelles. The growth law for the population density of gazelles becomes modified:

$$\frac{dn}{dt} = 15n^2 - 5n^3 - cn,$$

where c is the number of lions hunting gazelles each day. Given that $c \leq 11$, find the new equilibrium population densities. Identify the density n_s at which the population is in stable equilibrium.

Explain what would happen if more than 11 lions hunted gazelles each day.

Assuming $c \leq 11$, and the number of gazelles caught each day by each lion is proportional to n_s, show that $c = 10$ maximises the total number caught daily by the pride.

[15 marks]

14. The population densities of two interacting species evolve with time according to

$$\frac{dx}{dt} = x(6-4x) - 2xy, \quad \frac{dy}{dt} = y(1-2y) + xy,$$

where $x(t)$ and $y(t)$ are the population densities of the two species.

Comment on the biological significance of each of the terms on the right hand sides of these equations.

Show that there are equilibria at $(0,0)$, $(0,\frac{1}{2})$, $(\frac{3}{2},0)$, $(1,1)$ and classify the first three of these.

Linearise the growth equations in the neighbourhood of the coexistence equilibrium, $(1,1)$, using the substitutions $x = 1+\epsilon_x$, $y = 1+\epsilon_y$ where the deviations from equilibrium are small. Hence verify that the particular solutions corresponding to the initial conditions $x(0) = 1+\delta$, $y(0) = 1$ are given approximately by

$$x(t) = 1 + \delta e^{-3t}(\cos t - \sin t), \quad y(t) = 1 + \delta e^{-3t}\sin t.$$

What kind of equilibrium is $(1,1)$?

[15 marks]