M224 May 2003
Answers

1 The equation is separable, the general solution is

$$y = \frac{2}{\ln|x+3| - \ln|x+1| + C} .$$

2 The equation is homogeneous, the solution is

$$y = x\sqrt{4 + 6\ln|x|} .$$

3 The solution of the system of equations is

$$\begin{aligned} x &= \frac{1}{4}\left(5 + 3e^{4t}\right) \\ y &= \frac{1}{2}\left(5 - 3e^{4t}\right) . \end{aligned}$$

4(i) Start with the definition of the Laplace transform,

$$F(s) = \mathcal{L}\left(f(t)\right) = \int_0^\infty e^{-st} f(t)\, dt .$$

Take the derivative of both sides with respect to s

$$\begin{aligned} \frac{d}{ds}F(s) &= \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt = -\int_0^\infty e^{-st} t f(t)\, dt \\ \Rightarrow \quad \mathcal{L}\left(t f(t)\right) &= -\frac{d}{ds}F(s) . \end{aligned}$$

4(ii) The easiest way is to use the result of the first part. Start from $\mathcal{L}\left(e^{-2t}\right) = (s+2)^{-1}$ and apply the result of 4(i) twice. The result is

$$\mathcal{L}\left(t^2 e^{-2t}\right) = \frac{2}{(s+2)^3} \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{(s+2)^3}\right) = \frac{1}{2}t^2 e^{-2t} .$$

Another method is to use the result $\mathcal{L}(t^2) = 2s^{-3}$ and the “shift theorem”.

5(i) The characteristic curves are given by

$$ye^{-2x^2} = C$$

where C is a constant.
5(ii) The solution satisfying the boundary condition is

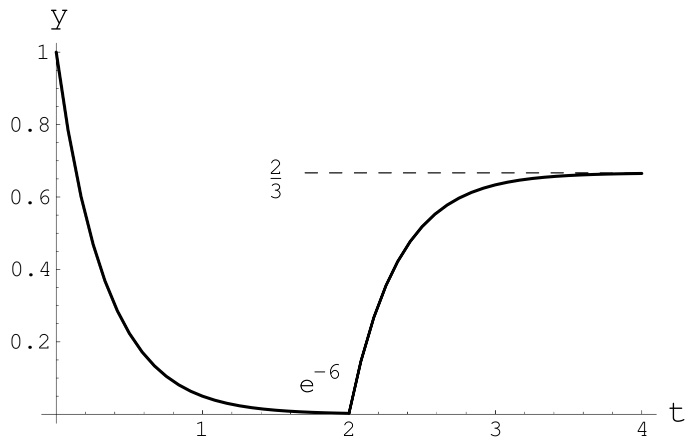
$$u(x, y) = \sin\left(ye^{-2x^2}\right) .$$

6(i) Standard book-work - see the lecture notes.
6(ii) The Laplace transform of y is given by

$$Y(s) = \frac{1}{s+3} + \frac{2}{s(s+3)}\,e^{-2s} .$$

Use the result of part (i) to find y

$$y = e^{-3t} + \frac{2}{3}H(t-2) - \frac{2}{3}H(t-2)e^{-3(t-2)} .$$



The sketch should look like this – note that there is no jump in the value of y at $t = 2$, just a change in the slope.

7(i) The Cauchy-Riemann conditions are

$$u_x = v_y \qquad u_y = -v_x \ .$$

Does $u = 4\tan^{-1}(y/x)$ obey Laplace's equation? Use the chain rule and the hint to calculate the derivatives of u:

$$u_x = -\frac{4y}{x^2+y^2}, \qquad u_y = \frac{4x}{x^2+y^2},$$
$$u_{xx} = \frac{8xy}{(x^2+y^2)^2}, \qquad u_{yy} = -\frac{8xy}{(x^2+y^2)^2},$$

so

$$u_{xx} + u_{yy} = 0.$$

(A surprising number of candidates had trouble finding u_x and u_y. Be sure you know how to use the chain rule.)

7(ii) The conjugate harmonic function is

$$v(x,y) = -2\ln(x^2+y^2) + C.$$

8(i) The solution is

$$y(t) = \frac{3}{2}e^t - \frac{1}{2}e^{-t} + e^{-2t}.$$

8(ii) Using the Laplace transform method gives

$$Y(s) = \frac{2s^2+4s+3}{(s-1)(s+1)(s+2)} = \frac{3}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{s+2} \ .$$

Taking the inverse Laplace transform gives

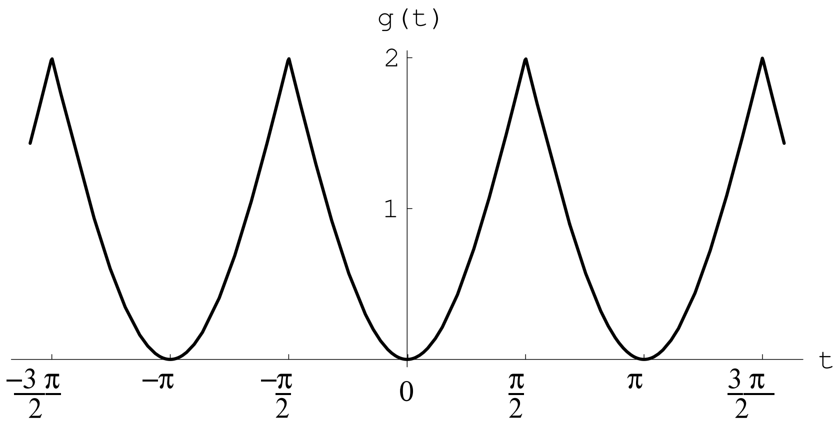
$$y(t) = \frac{3}{2}e^t - \frac{1}{2}e^{-t} + e^{-2t}$$

(which agrees with part (i)).

9 The general solution is

$$\begin{aligned} x &= Ae^{-2t} + Be^{6t} + \frac{1}{6} + \frac{1}{5}e^{t} \\ y &= 2Ae^{-2t} - 2Be^{6t} + \frac{2}{3} + \frac{3}{5}e^{t} \,. \end{aligned}$$

10(i) The period is π. The sketch should look like this.



10(ii) The Fourier series for $2 - 2|\cos t|$ is

$$g(t) = 2 - \frac{4}{\pi} + \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}\cos(2nt).$$

11(i) Standard book-work, see lecture notes.
11(ii) The boundary conditions are

$$\begin{aligned} \sum_n \sin\left(\frac{n\pi y}{b}\right)\left[C_n \cosh\left(\frac{-n\pi a}{b}\right) + D_n \sinh\left(\frac{-n\pi a}{b}\right)\right] &= -1 \\ && (\star) \\ \sum_n \sin\left(\frac{n\pi y}{b}\right)\left[C_n \cosh\left(\frac{n\pi a}{b}\right) + D_n \sinh\left(\frac{n\pi a}{b}\right)\right] &= 1\,. \end{aligned}$$

Taking the sum of these equations soon gives

$$C_n = 0.$$

To find D_n, write both sides of ($\star$) as Fourier series, the result is

$$D_n \sinh\left(\frac{n\pi a}{b}\right) = \begin{cases} 0 & \text{if } \quad n \quad \text{even} \\ \frac{4}{n\pi} & \text{if } \quad n \quad \text{odd} \end{cases}$$

so the final solution is

$$u(x,y) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin\left[(2k+1)\frac{\pi y}{b}\right] \sinh\left[(2k+1)\frac{\pi x}{b}\right]}{(2k+1)\sinh\left[(2k+1)\frac{\pi a}{b}\right]}.$$

A common mistake was to forget the sum $\sum_n$ in ($\star$), and to try satisfying the boundary conditions for each individual n, which is impossible.

12(i) The boundary condition in a ring says that

$$u(\theta + 2\pi, t) = u(\theta, t).$$

Repeat the standard arguments for a separable equation using this boundary condition, (see lecture notes), and you should get the given result.
12(ii) Get the coefficients a_n, b_n by finding the Fourier series for the temperature distribution at $t = 0$.

$$\begin{aligned} b_n &= 0 \\ a_0 &= 50 \\ a_n &= \frac{200}{n\pi}\sin\left(\frac{n\pi}{4}\right) \qquad \text{for} \quad n > 0 \end{aligned}$$

The temperature distribution at time t is therefore

$$u(\theta, t) = 25 + \sum_{n=1}^{\infty} \frac{200}{n\pi}\sin\left(\frac{n\pi}{4}\right)\cos(n\theta)e^{-\kappa n^2 t}.$$