Answer all questions from section A. Your best three answers to section B will be taken into account.

SECTION A

1. In a game for two players, A and B, simultaneously the players show 1, 2, 3, or 4 fingers. If the sum of the numbers of fingers shown is even then player A wins, otherwise player B wins. The amount won is equal to the total number of fingers shown by both players. Calculate the outcome matrix for A.

Considering pure strategies, find the best security level for each player and show that there is no equilibrium play.

[5 marks]

2. There are two players and, at the start, two empty cups on the table in front of them. Player A has two balls coloured Amber and player B has two balls coloured Blue. In turn the players place one of their balls in either cup but not more that a total of two balls are allowed in a cup. If either player can arrange to have two of their own balls in a cup, then that player wins and the game ends, otherwise the game is drawn. Sketch a game tree, assuming player A goes first.

Show that the second player cannot win.

[6 marks]

3. Define a saddle point for a two-person zero-sum game played with pure strategies, where A's outcome matrix has entry u_{ij} corresponding to strategy A_i for A and B_j for B.

Show that, if both (A_i, B_j) and (A_k, B_l) are saddle points, then $u_{ij} = u_{kl}$.

Find all the saddle points of the matrix

$$\begin{pmatrix} -1 & -2 & -1 \\ 2 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

[5 marks]

4. The language AB over the alphabet $\Sigma \equiv \{a, b\}$ is defined by the regular expression

$$(ab)^*(a \vee b)$$

(i) Give all words from AB of length 3 or less.

(ii) Give two words from AB which are not in a^*b^*.

(iii) Give a recursive definition of AB.

(iv) Draw a transition diagram for a finite automaton which accepts only the words of AB (reading from the right and terminated by blank).

[10 marks]

5. Let Γ be the directed graph whose adjacency matrix is as shown below:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Thus there is a directed edge from c to b but no directed edge from b to c. Determine the number of directed walks of length 2 from d to f and write them down. Which of these are directed paths? Calculate the in-degree and the out-degree at each vertex of Γ. Find a directed Euler walk in Γ, and write down, in order, the vertices passed in this walk.

[9 marks]

6. Carry out a critical path analysis for the activity given below. Find the minimum time required to complete the task. Write down the critical path as a sequence of the α_i's. For each α_i not in this path, determine the float time.

Activity	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}
Time	7	4	5	6	12	18	5	9	10	3	15	11	14	2
Prerequisites			α_1	α_2	α_2	α_1	α_3	α_3	α_2	α_5	α_5	α_6	α_5	α_9
							α_4	α_4		α_8	α_8	α_7	α_8	α_{11}
												α_{10}		

[10 marks]

7. (i) Using the method of sentence tableau, decide whether or not the following sequent is valid:

$$\neg(p \wedge q) \longrightarrow (r \longrightarrow \neg q) \quad \vdash \quad (r \longrightarrow p) \vee \neg q$$

(ii) Prove that a disjunctive normal form for $(p \vee q) \longrightarrow p$ is $p \vee \neg q$. Hence prove that a disjunctive normal form for

$$\big(p \longrightarrow (p \vee q)\big) \wedge \big(\neg p \longleftrightarrow \neg q\big) \wedge \big((p \vee \neg q) \longrightarrow p\big)$$

is $p \wedge q$. [10 marks]

SECTION B

8. Solve the following two-player zero-sum games, giving the optimal strategy for each player and the value of the game:

(i)

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

(ii)

$$\begin{pmatrix} 0 & -2 & -1 & 2 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

(iii)

$$\begin{pmatrix} 0 & 1 & 1 & -2 \\ 0 & -1 & -2 & 1 \end{pmatrix}$$

[15 marks]

9. Consider the languages X and Y over the alphabet $\Sigma \equiv \{a, b\}$ which contain the set of words of the form

$$\text{X}: \quad a^n(ab)^m$$

$$\text{Y}: \quad a^n(ab)^n$$

where $n,\ m$ are non-negative integers.

Give productions for each of these languages and discuss the classification of the grammar in each case.

Describe two machines: one which accepts only words of the language X and the other which accepts only words of the language Y. You may assume in each case that the machine starts at the rightmost element of the word and that the word is terminated by a blank at the left.

[15 marks]

10. Consider the following three sentences:

(a) $\neg p \longleftrightarrow (r \longrightarrow \neg q)$

(b) $r \longrightarrow \big((p \wedge q) \vee (r \longleftrightarrow \neg r)\big)$

(c) $(p \vee \neg r) \longrightarrow q$

Prove that none of these sentences implies either of the others. Show also that precisely one of the sentences is false under every assignment of truth values to the propositional variables making both of the other two sentences false.

[15 marks]

11. (i) Use the method of sentence tableau to determine whether or not the following argument is valid:

$$\exists x\Big(R(x) \longrightarrow \exists y(P(x,y) \longrightarrow \neg Q(y))\Big)$$

$$\forall y\Big(Q(y) \vee \forall x(\neg S(x) \longrightarrow P(x,y))\Big)$$

$$\vdash \exists x(R(x) \longrightarrow S(x))$$

(ii) Write the following argument symbolically and decide whether or not it is valid.

When it snows some people go skiing;
in summer, there are no skiers;
therefore it does not snow in summer.

[15 marks]