SECTION A

1. State the range of the function

$$f(x) = \cos(x).$$

Sketch the graph of $y = f(x)$ for $-2\pi \leq x \leq 2\pi$, indicating the values of x where it crosses the x-axis.

[4 marks]

2. By evaluating $f(0)$, $f'(0)$, and $f''(0)$, obtain the Maclaurin series expansion of the function

$$f(x) = \sqrt{1 + 2x}$$

up to and including the term in x^2.

[5 marks]

3.

a) Convert $(x, y) = (-\sqrt{2}, \sqrt{2})$ from Cartesian to polar coordinates.

b) Convert $(r, \theta) = (2, 3\pi/2)$ from polar to Cartesian coordinates.

In both parts, full marks will only be obtained if exact answers are given in terms of π, $\sqrt{2}$, etc.

[6 marks]

4. Calculate the integral

$$\int_0^{\pi/4} \left(x^2 - \sin(2x)\right) \, dx,$$

evaluating the result to three decimal places.

[5 marks]

5. Which of the following limits exist? Give the value of those which do.

$$\text{a) } \lim_{x\to 0} \sin(3x); \qquad \text{b) } \lim_{x\to 0} \frac{\sin(3x)}{x}; \qquad \text{c) } \lim_{x\to\infty} \sin(3x).$$

[8 marks]

6. Differentiate the following functions with respect to x:

$$\text{a) } x\cosh x; \qquad \text{b) } \sin(1+x^2); \qquad \text{c) } \frac{e^x}{x}.$$

[6 marks]

7. Show that the function

$$f(x) = x^3 - 3x^2 - 24x + 1$$

has exactly two stationary points. Determine whether each of these stationary points is a local maximum, a local minimum, or a point of inflection.

[5 marks]

8. Let z_1 and z_2 be the complex numbers given by $z_1 = 2 - j$ and $z_2 = 1 + j$. Calculate $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$, and z_1/z_2.

[6 marks]

9. State the value of $\sin^{-1}(1/2)$ (you should give an exact answer in radians). Give the general solution of the equation

$$\sin\theta = \frac{1}{2}.$$

[4 marks]

10. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Find $\mathbf{a} + \mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $|\mathbf{a}|$, $|\mathbf{b}|$, and $\mathbf{a} \cdot \mathbf{b}$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$ (in radians, to three decimal places)?

[6 marks]

SECTION B

11. Give the Maclaurin series expansion of the function $f(x) = e^x$ up to and including the term in x^4 (you are not required to show any working if you remember this expansion).

[2 marks]

Hence, or otherwise, determine the Maclaurin series expansions of the following functions, up to and including the terms in x^4:

a) xe^x; b) e^{2x}; c) $(e^x)^2$; d) $e^{(x^2)}$.

[10 marks]

Use the Maclaurin series expansion of e^x up to the term in x^4 to obtain an approximation to $e^{0.1}$. You should give your approximation to 6 decimal places.

[3 marks]

12. Calculate the radius of convergence R of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+4}\, x^n.$$

[8 marks]

Use the alternating series test to show that the series converges when $x = R$. Write down the series when $x = -R$, and state whether it is convergent or divergent. Hence state all of the (real) values of x for which the power series converges.

[7 marks]

13. By sketching the graphs of $y = x^2$ and $y = e^x$ on the same axes, explain why the equation

$$f(x) = x^2 - e^x = 0$$

has exactly one solution with $x \leq 0$ (you are not required to consider whether there are any solutions with $x > 0$).

[7 marks]

Use the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

with an initial guess $x_0 = -0.8$ to obtain successive approximations x_1, x_2, and x_3 to this solution. You should give each approximation to 6 decimal places.

[8 marks]

14. Let $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 + x & \text{if } x \leq 0 \\ \frac{x}{x-2} & \text{if } x > 0. \end{cases}$$

Sketch the graph of $y = f(x)$, indicating clearly the positions of any zeros, stationary points, and asymptotes. (You will get no marks for sketching the graph unless you show how you have determined the positions of these features.)

[12 marks]

At which value or values of x is $f(x)$ not continuous? At which value or values is it not differentiable?

[3 marks]

15.

By using de Moivre's theorem, find the integers a, b, c, d, e, and f such that

$$\begin{aligned} 8\cos^4\theta &= a\cos 4\theta + b\cos 2\theta + c \qquad \text{and} \\ 16\sin^5\theta &= d\sin 5\theta + e\sin 3\theta + f\sin\theta. \end{aligned}$$

[11 marks]

Check both of these results when $\theta = \frac{\pi}{2}$.

[4 marks]