SECTION A

1. Given $G = 6.67259 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, $M = 5.97370 \times 10^{24}$ kg and $R = 6.37814 \times 10^6$, calculate $\frac{GM}{R^2}$ to an accuracy to 5 significant figures.

[4 marks]

2. Simplify

(i)
$$\frac{(p^2q^3)^2}{(q^2r^3)^3(rp^{-2})^4},$$

(ii)
$$\frac{1}{2}\left[\frac{1}{x-\sqrt{7}} - \frac{1}{x+\sqrt{7}}\right].$$

[5 marks]

3. Find 2 values of θ between 0 and 2π which satisfy $\tan\theta = 0.76237$.
Give your answers in radians to 3 significant figures. [3 marks]

4. Using $\sin(A+B) = \sin A\cos B + \cos A\sin B$, show that

$$\sin\left(\frac{5\pi}{12}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

[4 marks]

5. Evaluate the sum of the geometric series

$$\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n.$$

Write down the first four terms of this series. [4 marks]

6. An arithmetic series has 30 terms. The first term is -27 and the common difference is 3. Find (i) the value of the last term and (ii) the sum of the series.

[4 marks]

7. Solve the linear equations

$$4x + 3y = 5, \qquad 3x - 4y = 10.$$

[4 marks]

8. By completing the square, find the two roots of the equation

$$x^2 - 6x - 91 = 0.$$

[5 marks]

9. Find the second derivative with respect to x of $\exp(2x)\sin(2x)$.

[5 marks]

10. Evaluate

$$\int_1^3 (t-1)^9 \mathrm{d}t.$$

[4 marks]

11. Evaluate the integral

$$\int_1^3 \frac{6}{3x-2}\mathrm{d}x.$$

Express your answer as a single logarithm. [4 marks]

12. Find all the second order partial derivatives of $u(x,y) = x^4 + 2x^2y^2 + y^4$.
[4 marks]

13. Given the complex numbers $z_1 = 3 + 4j$ and $z_2 = 5 - 5j$, find $|z_1|$, $\arg(z_2)$, $z_1 - z_2$ and z_2/z_1. [5 marks]

SECTION B

14 (a). The series $S(x)$ is given by

$$S(x) = \sum_{r=0}^{6} \frac{6!}{r!(6-r)!} x^r.$$

Expand this series. Simplify your result.

[7 marks]

(b). Express $\cosh x$ and $\sinh x$ in terms of the exponential function.

Given that $\tanh x = \sinh x / \cosh x$, show that

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

Using this result show that if $x = \tanh^{-1} \frac{1}{2}$, then $x = \frac{1}{2} \ln 3$. [8 marks]

15 (a). Show that

$$\int_0^{\pi} \theta^2 \cos 2\theta \; d\theta = \frac{\pi}{2}.$$

[8 marks]

(b). Given that

$$\frac{(x+5)}{(x-1)(x+1)} \equiv \frac{A}{x-1} + \frac{B}{x+1},$$

find A and B.

Using this result, or otherwise, show that

$$\int_2^3 \frac{(x+5)}{(x^2-1)} dx = \ln(9/2).$$

[7 marks]

16. Given that $x = 6$ is a root of the cubic

$$y = \mathrm{f}(x) = x^3 - 6x^2 - 4x + a,$$

find the value of the constant a.

Hence, or otherwise, find the factors of $\mathrm{f}(x)$.

Find and classify the stationary points of $\mathrm{f}(x)$.

Find the point of inflection.

Sketch $\mathrm{f}(x)$. [15 marks]

17 (a). State de Moivre's theorem. Use this theorem to show that

$$\cos 4\theta = 1 - 8\cos^2\theta\sin^2\theta.$$

[7 marks]

(b). Find the six roots of $z^6 = 1$. Hence, or otherwise, show that

$$z^6 - 1 = (z-1)(z+1)(z^2 - z + 1)(z^2 + z + 1).$$

[8 marks]

18. Sketch the triangle, $\mathcal{R}$, with sides given by $x = 0$, $y = 0$ and $y = L - x$, where L is a constant.

The function $u(x, y)$ is defined by

$$u(x, y) = u_0 \sin\left(\frac{\pi x}{L}\right)\sinh\left(\frac{\pi y}{L}\right),$$

where u_0 is a constant.

Show that $u(x, y)$ is zero on two of the sides of this triangle.

Show that $u(x, y)$ satisfies Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

[8 marks]

Show that

$$\int\int_{\mathcal{R}} u(x, y)\mathrm{d}x\mathrm{d}y \approx 0.435u_0L^2.$$

[You may use $\int_0^L \sin(\frac{\pi x}{L})\cosh(\frac{\pi(L-x)}{L}) \approx 2.004L$.]

[7 marks]