PAPER CODE NO.
**MATH185**

THE UNIVERSITY
*of* LIVERPOOL

# JANUARY 2007 EXAMINATIONS

Bachelor of Science : Year 1
Master of Physics : Year 2

## MATHEMATICS FOR PHYSICS

TIME ALLOWED : Two Hours and a Half

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INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B.
Section A carries 55% of the available marks.

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## SECTION A

**1.** Simplify

$$\frac{(a^3b^2)^{1/4}(a^7c)^3}{(ab^2c^3)^2}\ .$$

**[4 marks]**

**2.** Sketch the graph of $y = 3\sin(3x)$ for $-\pi \leq x \leq \pi$.

**[3 marks]**

**3.** The sum of the arithmetic series

$$S_n \;=\; a + (a+d) + (a+2d) + (a+3d) + \ldots + (a+(n-1)d)$$

is given by

$$S_n \;=\; \tfrac{1}{2}n(2a + (n-1)d)\ .$$

Use this result to find the sum of the first 80 even numbers.

**[4 marks]**

**4.** The sum of the geometric series

$$S_n \;=\; a + ar + ar^2 + ar^3 + \ldots ar^{(n-1)}$$

is given by

$$S_n \;=\; \frac{a(1-r^n)}{(1-r)}\ .$$

Use this result to find the sum of the finite series

$$1 - 0.2 + 0.04 - 0.008\ .$$

Verify your result by explicitly summing the series. Find also the sum of the infinite series

$$1 - 0.2 + 0.04 - 0.008 + \ldots$$

to four decimal places.

**[6 marks]**

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**5.** Solve the pair of simultaneous equations

$$3x - 2y \ = \ 2 \ \ , \ \ x + 3y \ = \ -3 \ .$$

[**4 marks**]

**6.** Solve the quadratic equation

$$x^2 \ + \ 11x \ + \ 28 \ = \ 0 \ .$$

[**3 marks**]

**7.** Differentiate the following functions

$$\text{(a)} \quad e^{x+3} \qquad \text{(b)} \quad \cos(2x+5) \qquad \text{(c)} \quad \frac{e^{-x}}{(x+1)} \qquad .$$

[**6 marks**]

**8.** Evaluate the integral

$$\int_1^3 \frac{dx}{x^2} \ .$$

Sketch a graph of the function $\frac{1}{x^2}$ in $x \geq 0$ and indicate on the sketch the feature represented by the integral calculated above.

[**4 marks**]

**9.** By substituting $u = (x^2 + 4)$ evaluate the integral

$$\int_2^3 \frac{x}{(x^2+4)} \, dx \ .$$

[**5 marks**]

**10.** Consider the function

$$f(x,y) \;=\; \sin(x+y)\ .$$

Find the partial derivatives $f_x$, $f_y$, $f_{xx}$, $f_{xy}$ and $f_{yy}$.

**[5 marks]**

**11.** Let $z_1 = 1 + 4i$ and $z_2 = 2 - i$.
Express $z_1 z_2$ and $\frac{z_1}{z_2}$ in the form of $a + ib$ where $a$ and $b$ are real.
Find $|z_1 z_2|$ and $\left|\frac{z_1}{z_2}\right|$.

**[6 marks]**

**12.** The hyperbolic functions are defined by

$$\cosh x \;=\; \tfrac{1}{2}(e^x + e^{-x}) \quad \text{and} \quad \sinh x \;=\; \tfrac{1}{2}(e^x - e^{-x})\ .$$

From the definition show that

$$\cosh(-x) \;=\; \cosh x \quad \text{and} \quad \sinh(-x) \;=\; -\ \sinh x\ .$$

If $\sinh x = 2$ show that $x = \ln(2 + \sqrt{5})$.

**[5 marks]**

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## SECTION B

**13.** Consider the function

$$f(x) \;=\; \frac{(x+A)}{(x-2)} \;\;,\;\; x \;\neq\; 2$$

where $A$ is a constant and $A \neq -2$. Find the corresponding inverse function $f^{-1}(x)$.

[**3 marks**]

For the case $A = 3$ sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. Show that the curves cross each other and also the line $y = x$ when

$$x \;=\; \frac{3 \pm \sqrt{21}}{2} \;.$$

[**12 marks**]

**14.** (a) Using integration by parts determine the integral

$$\int xe^{-3x}dx \;.$$

[**5 marks**]

(b) Find $A$ and $B$ such that

$$\frac{(x-4)}{(x+2)(x-3)} \;=\; \frac{A}{(x+2)} \;+\; \frac{B}{(x-3)} \;.$$

Hence calculate

$$\int_1^2 \frac{(x-4)}{(x+2)(x-3)}dx \;.$$

[**10 marks**]

**15.** Verify that $x = -1$ is a solution of the equation $x^3 + 3x^2 - x - 3 = 0$ and hence find the other solution or solutions.

[**4 marks**]

Find and classify the stationary points of the function

$$f(x) \;=\; x^3 \;+\; 3x^2 \;-\; x \;-\; 3 \;.$$

[**6 marks**]

Find the inflection point of $f(x)$.

[**2 marks**]

Sketch the graph of $y = f(x)$.

[**3 marks**]

**16.** (a) De Moivre's theorem states that

$$(\cos\theta \ + \ i\sin\theta)^n \ = \ \cos(n\theta) \ + \ i\sin(n\theta) \quad \text{for} \ \ n \ = \ 0, \pm1, \pm2, \ldots \ .$$

Using the theorem, prove that if $\theta$ is real then

$$\sin(3\theta) \ = \ 3\sin\theta \ - \ 4\sin^3\theta \ .$$

Express $\cos(3\theta)$ in terms of $\cos\theta$.

[**8 marks**]

(b) Find all the complex numbers $z$ such that $z^4 = 1$, writing your answers both in modulus-argument form and in the form $a + ib$. Indicate their positions on an Argand diagram.

[**7 marks**]

**17.** Sketch the region $\mathcal{R}$ bounded by the $x$-axis, the curve $y = x^3$ and the line $x = 2$.
A lamina with shape $\mathcal{R}$ has a mass density

$$\rho \ = \ mx$$

where $m$ is a constant. The mass $M$ of the lamina is given by

$$M \ = \ \int_{\mathcal{R}} \rho(x,y)dA \ = \ m\int_0^2 \int_0^{x^3} xdydx \ .$$

Find $M$ in terms of $m$.

[**5 marks**]

The centre of mass of the lamina is given by $(X, Y)$ where

$$X \ = \ \frac{1}{M}\int_{\mathcal{R}} x\rho(x,y)dA \quad \text{and} \quad Y \ = \ \frac{1}{M}\int_{\mathcal{R}} y\rho(x,y)dA \ .$$

Show that $X = \frac{5}{3}$ and find $Y$.

[**10 marks**]