MATH185 JAN 2006

Instructions to candidates

Answer all of section A and THREE questions from section B. Section A carries 55% of the available marks.

SECTION A

1. Simplify

$$\frac{(xy^3)^{\frac{1}{3}}(x^{\frac{1}{4}}z^3)^4}{xy^3z^2}$$

[4 marks]

2. Sketch the graph of $y = 2\sin 2x$ for $-\pi \leq x \leq \pi$.

[3 marks]

3. The sum of the arithmetic series

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \cdots + (a+(n-1)d)$$

is given by

$$S_n = \frac{n}{2}(2a + (n-1)d).$$

Use this result to find the sum of the first 50 even numbers. [4 marks]

4. The sum of the geometric series

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{(n-1)}$$

is given by

$$S_n = \frac{a(1-r^n)}{1-r}$$

Use this result to find the sum of the finite series

$$1 + 0.8 + 0.64 + 0.512.$$

Verify your result by explicitly summing the series. Find also the sum of the infinite series

$$1 + 0.8 + 0.64 + 0.512 + \cdots.$$

[6 marks]

5. Solve the pair of simultaneous equations

$$x + 2y = 3, \quad 3x + 4y = -1.$$

[4 marks]

6. Solve the quadratic equation

$$x^2 + 11x + 24 = 0.$$

[3 marks]

7. Differentiate the following functions:

$$(a) \quad x \tan x \qquad (b) \quad \frac{\sin x}{x+1} \qquad (c) \quad e^{x^2}$$

[6 marks]

8. Calculate the integral $\int_0^{\frac{\pi}{2}} \cos x \, dx$.

Sketch a graph of the function $y = \cos x$ and indicate on the sketch the feature represented by the integral calculated above.

[4 marks]

9. Use integration by parts to evaluate the integral

$$\int_0^{\pi} x \sin x \, dx$$

[5 marks]

10. Consider the function

$$f(x, y) = x \sin y + y \cos x.$$

Find the partial derivatives f_x, f_y, f_{xx}, f_{yy} and f_{xy}. [5 marks]

11. Let $z_1 = 3 + 5i$ and $z_2 = 1 + i$.

Express $z_1 z_2$ and $\frac{z_1}{z_2}$ in the form $a + ib$ where a and b are real.

Find $|z_1 z_2|$ and $|\frac{z_1}{z_2}|$

[6 marks]

12. The hyperbolic functions are defined by

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \quad \text{and} \quad \sinh x = \frac{1}{2}(e^x - e^{-x}).$$

Sketch the graphs of $y = \cosh x$ and $y = \sinh x$.

From the definitions, prove that

$$\cosh^2 x - \sinh^2 x = 1 \quad \text{and} \quad \cosh^2 x + \sinh^2 x = \cosh 2x$$

[5 marks]

SECTION B

13. Consider the function

$$f(x) = \frac{x + A}{x + 1}, \quad x \neq -1.$$

where A is a constant, ($A \neq 1$). Find the corresponding inverse function $f^{-1}(x)$.

[3 marks]

For the case $A = 2$, sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. Show that the the curves cross each other and also the line $y = x$ when

$$x = \pm\sqrt{2}.$$

[12 marks]

14. (a) Using the substitution $u = \tan x$, or otherwise, evaluate the integral

$$\int \sec^2 x \sqrt{\tan x}\, dx$$

[4 marks]

(b) Find constants A, B, such that

$$\frac{x - 2}{(x + 1)(x + 2)} \equiv \frac{A}{x + 1} + \frac{B}{x + 2}.$$

Hence or otherwise show that

$$\int_0^2 \frac{x - 2}{(x + 1)(x + 2)}\, dx = 4 \ln 2 - 3 \ln 3.$$

[11 marks]

15. Verify that $x = 2$ is a solution of the equation $x^3 - 9x^2 + 24x - 20 = 0$, and hence find the other solution or solutions.

[3 marks]

Find and classify the stationary points of the function

$$f(x) = x^3 - 9x^2 + 24x - 20.$$

[7 marks]

Find the inflection point of $f(x)$.

[2 marks]

Sketch the graph $y = f(x)$.

[3 marks]

16. (a) De Moivre's theorem states that

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx, \quad \text{for} \quad n = 0, \pm 1, \pm 2, \cdots$$

Using the theorem, prove that if θ is real then

$$\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta.$$

[8 marks]

(b) Find all complex numbers z such that $z^3 = 1$, writing your answers both in modulus-argument form and in the form $a + ib$. Indicate their positions in an Argand diagram.

[7 marks]

17. Sketch the region $\mathcal{R}$ bounded by the x-axis, the curve $y = 1/x$, and the lines $x = 1$ and $x = 2$.

A lamina with shape $\mathcal{R}$ has a mass density given by

$$\rho = mx,$$

where m is a constant. The mass M of the lamina is given by

$$M = \int_{\mathcal{R}} \rho(x, y)\, dA = m \int_1^2 \int_0^{1/x} x\, dy\, dx.$$

Find M in terms of m.

[5 marks]

The centre of mass of the lamina is given by (X, Y) where

$$X = \frac{1}{M} \int_{\mathcal{R}} x\rho(x, y)\, dA \quad \text{and} \quad Y = \frac{1}{M} \int_{\mathcal{R}} y\rho(x, y)\, dA.$$

Find X and Y.

[10 marks]