MATH185 Jan 2005

Instructions to candidates

Answer all of section A and THREE questions from section B. Section A carries 55% of the available marks.

SECTION A

1. Simplify

$$\frac{(x^2y^3)^{\frac{1}{3}}(x^{\frac{1}{4}}z^2)^4}{xy^2z^3}$$

[4 marks]

2. Sketch the graph of $y = \sin 2x$ for $-\pi \leq x \leq \pi$.

[3 marks]

3. The sum of the arithmetic series

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \cdots + (a + (n-1)d)$$

is given by

$$S_n = \frac{n}{2}(2a + (n-1)d).$$

Use this result to find the sum of the first 100 even numbers. [4 marks]

4. The sum of the geometric series

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{(n-1)}$$

is given by

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Use this result to find the sum of the finite series

$$1 - 0.3 + 0.09.$$

Verify your result by explicitly summing the series. Find also the sum of the infinite series

$$1 - 0.3 + 0.09 - 0.027 + \cdots.$$

(Give your answer in the form of a fraction). [6 marks]

5. Solve the pair of simultaneous equations

$$2x + 3y = 4, \quad 3x + 5y = 8.$$

Check your answer by substituting back into the original equations. [5 marks]

6. Solve the quadratic equation

$$2x^2 + x - 3 = 0.$$

[3 marks]

7. Differentiate the following functions:

$$(a) \quad e^{4x} \qquad (b) \quad \sin(x^2) \qquad (c) \quad x \cos x.$$

[6 marks]

8. Calculate the integral $\int_2^3 \frac{1}{x}\,dx$, evaluating the result to three decimal places.

Sketch a graph of the function $y = \frac{1}{x}$ and indicate on the sketch the feature represented by the integral calculated above.

[5 marks]

9. Use integration by parts to evaluate the integral

$$\int_0^\infty xe^{-x}\,dx$$

[4 marks]

10. Consider the function

$$f(x, y) = \frac{1}{x + 2y}.$$

Find the partial derivatives f_x, f_y, f_{xx}, f_{yy} and f_{xy}. [5 marks]

11. Let $z_1 = 1 + 2i$ and $z_2 = 2 - i$.

Express $z_1 z_2$ and $\frac{z_1}{z_2}$ in the form $a + ib$ where a and b are real.

Find $|z_1 z_2|$ and $|\frac{z_1}{z_2}|$

[6 marks]

12. The hyperbolic functions are defined by

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \quad \text{and} \quad \sinh x = \frac{1}{2}(e^x - e^{-x}).$$

From the definitions, prove that

$$\frac{d}{dx}\sinh x = \cosh x \quad \text{and} \quad \frac{d}{dx}\cosh x = \sinh x.$$

[4 marks]

SECTION B

13. Consider the function

$$f(x) = \frac{Ax+1}{x-1}, \quad x \neq 1.$$

where A is a constant. Find the corresponding inverse function $f^{-1}(x)$.

[4 marks]

(i) For the case $A = 2$, sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. Show that the the curves cross each other and also the line $y = x$ when

$$x = \frac{3 \pm \sqrt{13}}{2}.$$

[9 marks]

(ii) Find a different value of A such that $f(x)$ and $f^{-1}(x)$ are identical.

[2 marks]

14. (a) Using the substitution $u = x^2 + x + 3$, or otherwise, evaluate the integral

$$\int (2x+1)\sqrt{x^2+x+3}\, dx$$

[6 marks]

(b) Find constants A, B, such that

$$\frac{x-3}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}.$$

Hence or otherwise calculate

$$\int_3^4 \frac{x-3}{(x-1)(x-2)}\, dx$$

[9 marks]

15. Verify that $x = 1$ is a solution of the equation $x^3 - x^2 - 5x + 5 = 0$, and hence find the other two solutions.

[2 marks]

Find and classify the stationary points of the function

$$f(x) = x^3 - x^2 - 5x + 5.$$

[8 marks]

Find the inflection point of $f(x)$.

[2 marks]

Sketch the graph $y = f(x)$.

[3 marks]

16. (a) De Moivre's theorem states that

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx, \quad \text{for} \quad n = 0, \pm 1, \pm 2, \cdots$$

Using the theorem, prove that if θ is real then

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad \text{and} \quad \sin 2\theta = 2 \sin \theta \cos \theta.$$

[5 marks]

(b) Find all complex numbers z such that $z^3 = 8$, writing your answers both in modulus-argument form and in the form $a + ib$. Indicate their positions in an Argand diagram.

[10 marks]

17. Sketch the region $\mathcal{R}$ bounded by the x-axis, the curve $y = x^2$, and the line $x = 1$.

A lamina with shape $\mathcal{R}$ has a mass density given by

$$\rho = mx,$$

where m is a constant. The mass M of the lamina is given by

$$M = \int_{\mathcal{R}} \rho(x, y)\, dA = m \int_0^1 \int_0^{x^2} x\, dy\, dx.$$

Find M in terms of m.

[5 marks]

The centre of mass of the lamina is given by (X, Y) where

$$X = \frac{1}{M} \int_{\mathcal{R}} x\rho(x, y)\, dA \quad \text{and} \quad Y = \frac{1}{M} \int_{\mathcal{R}} y\rho(x, y)\, dA.$$

Show that

$$(X, Y) = (\frac{4}{5}, \frac{1}{3}).$$

[10 marks]