Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

SECTION A

1. Given $h = 6.62608 \times 10^{-34}$ J s, $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}, $c = 2.99792 \times 10^{8}$ m s^{-1}, $m_e = 9.10939 \times 10^{-31}$ kg and $e = 1.60218 \times 10^{-19}$ C, find the value of the Bohr radius, $a_0 = h^2/(\pi \mu_0 c^2 m_e e^2)$, in standard form accurate to 5 significant figures.

[5 marks]

2. Simplify

$$\frac{(x^3y^2z)^{\frac{1}{2}}(xy^2z)^{-\frac{3}{2}}}{(yz^2)^{\frac{1}{2}}} .$$

[3 marks]

3. Sketch $f(\theta) = \cos\theta$ for $0 \le \theta \le \pi$. On the same diagram sketch $f^{-1}(\theta)$. Evaluate $f^{-1}(0.62543)$. [4 marks]

4. Evaluate, as rational numbers,

(i) $$1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots + \frac{256}{6561} ,$$

(ii) $$12 + \frac{35}{3} + \frac{34}{3} + 11 + \cdots + \frac{26}{3} .$$

[7 marks]

5. Expand and simplify $(a + 2b)^5$. [4 marks]

6. By factorisation, or otherwise, find the roots of the equation

$$3x^2 - 2x - 21 = 0 .$$

[3 marks]

7. Draw the two straight lines

$$x - 2y + 3 = 0, \qquad 4x + 3y - 10 = 0 .$$

Find their point of intersection. [6 marks]

8. Find the second derivative with respect to x of $\ln(x^2 - 4)$.

[4 marks]

9. Evaluate

$$\int_2^4 (t^2 - 4t + 4)\ \mathrm{d}t\ .$$

[4 marks]

10. By substituting $u = 25 - x^2$, or otherwise, show that

$$\int_3^4 \frac{x}{25 - x^2}\ \mathrm{d}x = \ln\frac{4}{3}\ .$$

[5 marks]

11. Show that $u(x,t) = \sin(k(x - ct))$, where k and c are constants, satisfies the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\ .$$

[4 marks]

12. Given the complex numbers $z_1 = 4 - 3j$ and $z_2 = 5 + 5j$, find $|z_1|$, $z_2 + 2z_1$ and $1/z_1 + 1/\bar{z}_1$. [6 marks]

SECTION B

13. The hyperbolic functions sinh and cosh are defined by

$$\sinh(x) = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}), \qquad \cosh(x) = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) \ .$$

Find the derivatives of each of these functions, expressing your answer in terms of hyperbolic functions.

Show that

$$\cosh^2(x) - \sinh^2(x) = 1, \qquad \cosh^2(x) + \sinh^2(x) = \cosh(2x) \ .$$

and

$$2\sinh(x)\cosh(x) = \sinh(2x) \ .$$

[10 marks]

Given that $\cosh(x) = 2$ and $x > 0$, show that $x = \ln(2 + \sqrt{3})$ and $\cosh(2x) = 7$.

[5 marks]

14 (a). Evaluate the integral

$$\int_0^1 t^2 \mathrm{e}^{2t} \ \mathrm{d}t \ ,$$

expressing your answer in terms of e. [10 marks]

(b). Evaluate the integral

$$\int_1^\infty \frac{1}{1+x^2} \ \mathrm{d}x \ .$$

[5 marks]

15. Find and classify the stationary points of

$$y = f(x) = x^3 - 2x^2 - 5x + 6 \ .$$

Find the point of inflection. [8 marks]

Find the factors of $f(x)$.

Sketch $f(x)$. [7 marks]

16 (a). Use de Moivre's theorem to find $\sin 3\theta$ and $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Hence deduce that

$$\tan 3\theta = \tan\theta \frac{(3-\tan^2\theta)}{(1-3\tan^2\theta)} \ .$$

[7 marks]

(b). Show that $e^{2k\pi j} = 1$ for integer k.

Express $8\sqrt{3}+8j$ in modulus-argument form.

Hence find the four roots (in modulus-argument form) of

$$z^4 = 8\sqrt{3}+8j \ .$$

Show these roots as points on the Argand diagram. [8 marks]

17. Sketch the two regions

$\mathcal{R}_1 \ \{0 \le x \le 2, \ 0 \le y \le x\}$ and $\mathcal{R}_2 \ \{2 \le x \le 4, \ 0 \le y \le 4-x\}$.

For a point (x, y) in the combined region $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$,

(i) show that $0 \le y \le 2$,

(ii) find the inequalities satisfied by the $x-$coordinate. [5 marks]

Evaluate the double integral

$$\int\int_{\mathcal{R}} y(x+y) \ \mathrm{d}x\mathrm{d}y \ .$$

[10 marks]