SECTION A

1. Find all solutions (if there are any) of the systems of equations

(i)
$$\begin{aligned} x - 2y + 3z &= 1 \\ 2x - 4y + 4z &= 2 \\ -3x + 6y - 5z &= 1 \end{aligned}$$

(ii)
$$\begin{aligned} x - 2y + 3z &= 1 \\ 2x - 4y + 4z &= 2 \\ -3x + 6y - 5z &= -3. \end{aligned}$$

[11 marks]

2. (a) Find the greatest common divisor d of 533 and 1677, and find integers s and t such that

$$d = 533s + 1677t.$$

(b) Find the inverse of 25 mod 79.

[11 marks]

3. Find all solutions (if any) of each of the following congruences

(i) $10x \equiv 25 \bmod 43$.

(ii) $10x \equiv 25 \bmod 44$.

(iii) $10x \equiv 25 \bmod 45$.

[11 marks]

4. Let π and ρ be the permutations

$$\pi = (176)(3542), \quad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \end{pmatrix}.$$

Write $\pi^2, \pi\rho$ and ρ^{-1} as products of disjoint cycles and find the order and sign of each of these permutations. [11 marks]

5. Find the smallest positive number x which satisfies the two simultaneous congruences

$$x \equiv 4 \bmod 27 \quad \text{and} \quad x \equiv 5 \bmod 11.$$

Find also the next largest x which satisfies both congruences. [11 marks]

SECTION B

6. Find all eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & 0 \end{pmatrix}.$$

[15 marks]

7. Outline the method of coding and decoding messages using a public key code with base n and coding exponent a.

A code with base $221 = 13 \times 17$ and exponent 77 is used to encode a message, with letter - digit correspondence

T	H	E	L	A	Z	Y	D	O	G
0	1	2	3	4	5	6	7	8	9

Find the decoding exponent.

Hence decode the message 115/ 46 .

[15 marks]

8. Let G be a group. Say what it means for G to be *cyclic*.

Show that the group G_{15} of invertible congruence classes mod 15 is not cyclic.

Decide, giving reasons, which pairs of groups in the following list are isomorphic, where C_n denotes a cyclic group of order n.

$$C_4,\ C_2 \times C_2,\ G_{10},\ G_8.$$

[15 marks]

9. List the code words of the group code with generator matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

State how many errors are detected and how many are corrected by this code, giving reasons for your answers. Give a table of syndromes for this code for all possible single digit errors in transmission.

Suppose that the letters R U G B Y O D E are represented by the binary numbers 000, 001, 010, 011, 100, 101, 110, 111, respectively. Read the received message

$$0101001\ \ 1011010\ \ 1001010\ \ 1111100\ \ 0010110\ \ 1000101\ \ 1110001.$$

[15 marks]