PAPER CODE NO.
MATH 122

THE UNIVERSITY *of* LIVERPOOL

SEPTEMBER 2003 EXAMINATIONS

Bachelor of Science : Year 1
Bachelor of Science : Year 2
Master of Mathematics : Year 1
Master of Mathematics : Year 2
Master of Physics : Year 1

DYNAMICAL MODELLING

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.
Take $g = 9.81\text{ms}^{-2}$. Give numerical answers to 3 significant figures.
You may use

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

SECTION A

1. Six people join a queue at a MacDonald's fast food restaurant every two minutes and six people are served in the queue every three minutes. Write down a differential equation for $n(t)$, the number of people in the queue at time t, where the unit of time is one minute. How long does it take for there to be ten people in the queue, given initially there were none?

[5 marks]

2. Oil is pumped into a uniform cylindrical barrel of cross-sectional area 0.5m^2 at a constant rate of $1\times10^{-6}\,\text{m}^3\text{s}^{-1}$. There is a small hole in the bottom of the barrel and oil leaks out at a rate proportional to the height h of the oil in the barrel at time t. Show the differential equation for h is

$$\frac{dh}{dt} = 2\times10^{-6} - kh,$$

where k is a positive constant. Given that the height of the oil in the barrel approaches a maximum value of 0.5m, solve the differential equation to find how long it takes the oil to reach a height of 0.2m, given the barrel is initially empty.

[8 marks]

3. A son inherits his father's estate and the sum of £25000 on the latter's death. He invests this money in stock options, which over the years have yielded a steady 5% per annum. At the end of each year he withdraws 10% of the balance but invests a further £2400 profit he has made running the estate. Write down a discrete differential equation for u_m the value of his investment after m years, and determine its equilibrium value. How much is his investment worth after 5 years, and how long does it take for its value to reach £40000?

[6 marks]

4. If n the number of events that have occurred at time t, follows a Poisson process, then the probability $P(n,t)$ that n events have occurred by time t is given by

$$P(n,t)=\frac{(\lambda t)^n}{n!}e^{-\lambda t}.$$

At what time does $P(n,t)$ reach its maximum value? If, on average, there are 10 fatal road accidents on the M6 each year, calculate (assuming the number of accidents follows a Poisson process) the probability that this year only 8 fatal accidents occur. [5 marks]

5. A particle travels so that its velocity is given by

$$v=2t^2\,i+3\cos(t)\,j-3\sin(t)k\ .$$

Calculate at any time t a) the particle's speed, b) its acceleration, and c) its position, given that at $t=\pi$ seconds it was situated at $(0,4,-3)$.

[6 marks]

6. A shell is fired from a gun situated at the origin with a speed of $\sqrt{588.6}\text{ms}^{-1}$ at an angle of 30^0 to the horizontal. Neglecting air resistance, write down the equations of motion for the horizontal coordinate x and the vertical coordinate y of the shell at time t. Solve these equations to show that

$$y=\frac{15}{2}-\frac{\left(x-15\sqrt{3}\right)^2}{90}.$$

[10 marks]

7. A spring of negligible mass, when suspended vertically from one end, is stretched a distance of 0.2m when a mass m of 0.005kg is added. Calculate its spring constant λ. The un-stretched spring is placed horizontally on a rough table, with the mass attached to one end, the other being firmly fixed to a nearby wall. The mass is then pulled away from the wall a further 0.2m and released. Assuming friction imposes a damping force numerically equal to $14m$ times the instantaneous speed, show that the differential equation describing the motion of the mass is

$$\frac{d^2x}{dt^2}+14\frac{dx}{dt}+49x=0,$$

where x represents the extension of the spring (assume $g=9.8\text{ms}^{-2}$). Solve this equation and sketch $x(t)$ for any time $t>0$.

[9 marks]

8. Let $x(t) > 0$ and $y(t) > 0$ represent the levels of two populations governed by the following coupled differential equations

$$\frac{dx}{dt} = (y - 45) \qquad \frac{dy}{dt} = (30 - x).$$

Initially $x = 15$ and $y = 25$. Obtain and solve the differential equation for y in terms of x. From your results draw a phase diagram for this situation, indicating which way around the curve the point (x, y) moves.

[6 marks]

SECTION B

9. Consider a two-state stochastic system, with states A and B. In the usual notation,

$$\frac{d}{dt}P(A,t) = P(B,t)W(B \to A) - P(A,t)W(A \to B).$$

Write down what each term in this equation represents.

[5 marks]

In a local football league only wins or losses count (draws are decided by a penalty shoot out). The probability that Liverton FC win their next match after winning their previous one is 0.75. However, if they lose their previous match the probability they lose their next one is 0.6. If $P(W,t)$ and $P(L,t)$ are the probabilities of Liverton FC winning or losing their match at time t, show that

$$\frac{dP(W,t)}{dt} = 0.4 - 0.65P(W,t).$$

Solve this equation to find $P(W,t)$ given at the start of the season $P(W,0) = 0.5$. In the long term teams need to win 75% of their matches to become champions. Are Liverton FC likely to do this?

[10 marks]

10. A model for the number $x(t)$ of fleas per unit area at time t uses the equation

$$\frac{dx}{dt} = x^2 y,$$

where $y(t)$ is the number of rats per unit area at time t. The fleas give the rats a fatal disease. The number of rats is thought to satisfy the equation:

$$\frac{dy}{dt} = y - x^2 y.$$

Find the equation for dy/dx and integrate it, given that initially $x = 3/4$ and $y = 5/12$.

[6 marks]

Sketch the graph of y against x, indicating the realistic part of the graph and the direction of x and y change with time. Describe what happens to the two populations.

[9 marks]

11. Suppose the following differential equation

$$\frac{dn}{dt} = f(n),$$

has an equilibrium point at $n = N$ such that $f(N) = 0$. By considering what happens at $n = N + \varepsilon$, where ε is a small time dependent parameter, establish the conditions on $f(n)$ which determine the stability of the equilibrium point at $n = N$.

[5 marks]

Determine the equilibrium points of the following differential equation and their stability.

$$\frac{dn}{dt} = -n^2 + 15n - 50$$

Integrate the above equation (use partial fractions) to find $n(t)$ assuming $n = 7$, when $t = 0$. What happens to the value of n as $t \to \infty$?

[10 marks]

12. A particle of mass m is fired vertically upward from the origin $(y=0)$ at an initial speed v_0. Air resistance is directly proportional to the product of the mass of the particle and its instantaneous vertical speed, with constant of proportionality k. What are the units of k?
Write down a differential equation for the forces acting on the body, assuming $y>0$ represents the vertical distance travelled. By using the substitution $p=dy/dt$, or otherwise, integrate this equation to show the time taken for the particle to reach its maximum height is

$$t_{Max}=\frac{1}{k}\log_e\left[1+\frac{\mathrm{v}_0 k}{g}\right].$$

[9 marks]

Hence show the maximum height y_{Max} is given by

$$y_{Max}=\frac{\mathrm{v}_0}{k}-\frac{g}{k^2}\log_e\left[1+\frac{\mathrm{v}_0 k}{g}\right].$$

[6 marks]

13. At noon the pilot of a jumbo jet flying west at a speed of 600km/h is alarmed to observe from his cockpit window a fighter plane flying at the same height, some 1.5km due south of his air liner. It appears to be travelling northeast at 800km/h.
By using vector methods, find the actual velocity and speed of the fighter plane.

[6marks]

Find the position vector at time t of the fighter plane with respect to the jumbo.

[3 marks]

Find the distance of closest approach and the time, to the nearest second at which it occurs.

[6 marks]

[Use the unit vectors i and j to represent East and North respectively.]