Instructions to Candidates

You may attempy all questions. All answers to Section A and THREE answers from Section B will be taken into account.

SECTION A

1. Use differentiation to obtain the Taylor Polynomial of order two for $f(x) = \cos x$ about $x = 0$.

[4 marks]

2. Find the general solution of the differential equation

$$x\frac{dy}{dx} + 2y = 8x^2.$$

[4 marks]

3. Find the solution of the differential equation

$$y'' + 6y' + 5y = 0$$

that satisfies the boundary conditions $y(0) = 4$, $y'(0) = 0$.

[6 marks]

4. By choosing two different paths as $(x, y) \to (0, 0)$ or otherwise, show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + 4y^2}$$

does not exist.

[4 marks]

5. Let $z = u^2 - v^2$, where $u = xy$ and $v = x/y$. Use the chain rule to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of x and y

[4 marks]

6. The function $f(x,y,z)$ is defined to be

$$f(x,y,z) = xz - y^2 + z^3.$$

Find the gradient of f at the point $P = (1,-1,1)$, and the derivative of f at P in the direction of $(1,2,2)$.

Find the equation of the tangent plane to the surface $xz - y^2 + z^3 = 1$ at P.

[6 marks]

7. Find and classify any local maxima, minima or saddle points of

$$f(x,y) = 6y^2 - 8x^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4.$$

[8 marks]

8. Find a quadratic approximation to the function $f(x,y) = \dfrac{1}{x-y}$ valid near $(x,y) = (1,0)$.

[5 marks]

9. Evaluate the double integral

$$\int\int_R (x+2y)\,dx\,dy,$$

where R is the region bounded by the straight lines $x = 0$, $y = 0$ and $2x + y = 2$.

[6 marks]

10. By first changing the order of integration, evaluate the repeated integral

$$\int_0^2 \left(\int_{y/2}^1 y e^{x^3} dx \right) dy.$$

[8 marks]

SECTION B

11. a) Find the solution of the differential equation

$$(x^2 + xy)\frac{dy}{dx} = xy - y^2,$$

subject to the boundary condition $y(1) = 1$.

[8 marks]

b) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10x.$$

[7 marks]

12. Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = x^2y^2z^2$, subject to the constraint that the point (x, y, z) lies on the ellipsoid $x^2 + 4y^2 + 9z^2 = 27$.

[15 marks]

13. Given that $f(x, y) = \sqrt{4 - x^2 - y^2}$, find all first and second order partial derivatives of $f(x, y)$. Hence derive the Taylor series for $f(x, y)$ about $(x, y) = (0, 0)$ up to and including second order terms.

[11 marks]

Verify your answer by using the Binomial expansion of $(1 - z)^{1/2}$ and an appropriate substitution for z.

[4 marks]

14. A thin triangular plate occupies the region R bounded by the lines $x = 1$, $y = 0$ and $y = x$, and is of density $\rho(x, y) = x + y$. Show that the mass M satisfies

$$M = \int\int_R \rho dxdy = \frac{1}{2}.$$

Obtain the coordinates $(\overline{x}, \overline{y})$ of the centre of mass of the plate.

[15 marks]