

MATH102 Solutions September 2005
Section A

1. The Taylor series of $f(x) = e^{2x}$ is

$$1 + 2x + \frac{4x^2}{2} + \frac{8x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}.$$

This can be written down using the Taylor series of e^y at 0 with $y = 2x$, and can also be worked out directly by computing all derivatives of f at $x = 0$.

[3 marks]

- a) When $x = 1$ the series is convergent and equal to $f(1)$.

[1 mark]

- b) When $x = 100$ the series is also convergent and equal to $f(100)$. In fact the Taylor series is convergent and equal to $f(x) = e^{2x}$ for all x

[1 mark]

No explanation is required in a) or b).

5 = 3 + 1 + 1 marks

- 2 Separating the variables, we have

$$\int y dy = - \int x dx,$$

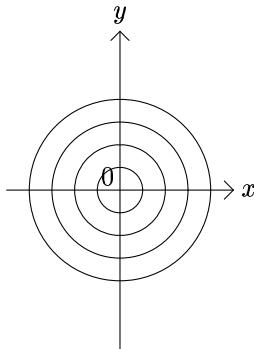
$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C,$$

or

$$y^2 + x^2 = 2C.$$

[2 marks]

The solution curves are circles centred on the origin of radius $\sqrt{2C}$, for any $C > 0$, as shown.



[3 marks]

For a solution with $y(1) = 1$ we have $1 + 1 = 2C$, that is $2C = 2$. So the solution satisfies $x^2 + y^2 = 2$. For y as a function of x , we have $y = \sqrt{2 - x^2}$, taking positive square root, since $y(1) = 1 > 0$.

[1 mark]

[2 + 3 + 1 = 6 marks]

3. Try $y = e^r x$. Then

$$r^2 - 4 = 0 \Rightarrow r = \pm 2.$$

So the general solution is

$$y = Ae^{2x} + Be^{-2x}.$$

[2 marks]

So $y' = 2Ae^{2x} - 2Be^{-2x}$ and the initial conditions $y(0) = 1$, $y'(0) = -1$ give

$$A + B = 1, \quad 2A - 2B = -1 \rightarrow 4A = 1, \quad A = \frac{1}{4}, \quad B = \frac{3}{4}.$$

So

$$y = \frac{1}{4}e^{2x} + \frac{3}{4}e^{-2x}.$$

[3 marks]

2 + 3 = 5 marks

4. We take limits along the lines $y = 0$ and $x = 0$. We have:

$$\lim_{y=0, x \rightarrow 0} \frac{x^2 + y^2}{x^2 + 2y^2} = \lim_{x \rightarrow 0} \frac{x^2}{x^2} = 1,$$

$$\lim_{x=0, y \rightarrow 0} \frac{x^2 + y^2}{x^2 + 2y^2} = \lim_{y \rightarrow 0} \frac{y^2}{2y^2} = \frac{1}{2}.$$

So we get different limits along different lines of approach, and the overall limit does not exist.

[4 marks]

5.

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \\ \frac{\partial f}{\partial y} &= -\frac{2xy}{(x^2 + y^2)^2}. \end{aligned}$$

[2 marks]

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{-2x}{(x^2 + y^2)^2} + \frac{4x(y^2 - x^2)}{(x^2 + y^2)^3} \\ &= \frac{2x^3 - 6xy^2}{(x^2 + y^2)^3}. \\ \frac{\partial^2 f}{\partial y^2} &= \frac{-2x}{(x^2 + y^2)^2} + \frac{8xy^2}{(x^2 + y^2)^3} \end{aligned}$$

$$= \frac{-2x^3 + 6xy^2}{(x^2 + y^2)^3}.$$

So

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

as required.

[4 marks]

[2 + 4 = 6 marks.]

6. By the Chain Rule,

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial F}{\partial u}(u, v) + \frac{\partial F}{\partial v}(u, v),$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial F}{\partial u}(u, v) - \frac{\partial F}{\partial v}(u, v),$$

So

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 F}{\partial u^2}(u, v) + \frac{\partial^2 F}{\partial v \partial u}(u, v) + \frac{\partial^2 F}{\partial u \partial v}(u, v) + \frac{\partial^2 F}{\partial v^2}(u, v),$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 F}{\partial u^2}(u, v) - \frac{\partial^2 F}{\partial v \partial u}(u, v) - \frac{\partial^2 F}{\partial u \partial v}(u, v) + \frac{\partial^2 F}{\partial v^2}(u, v),$$

and so

$$\frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) = 2 \left(\frac{\partial^2 F}{\partial u^2}(u, v) + \frac{\partial^2 F}{\partial v^2}(u, v) \right),$$

as required.

[6 marks]

7. For

$$f(x, y, z) = xy^2 + yz^2 - 2xyz,$$

we have

$$\nabla f(x, y, z) = (y^2 - 2yz)\mathbf{i} + (2xy + z^2 - 2xz)\mathbf{j} + (2yz - 2xy)\mathbf{k}$$

[2 marks] So

$$\nabla f(x, y, z)(1, 2, 1) = 3\mathbf{j}.$$

The tangent plane at (1,2,1) is

$$3(y - 2) = 3y - 6 = 0.$$

[3 marks]

[3 + 2 = 5 marks.]

8. For

$$f(x, y) = 2x^2 - 2x^2y + y^2,$$

we have

$$\frac{\partial f}{\partial x} = 4x - 4xy, \quad \frac{\partial f}{\partial y} = -2x^2 + 2y.$$

[2 marks]

So at a stationary point,

$$x(1-y) = 0 = 2x^2 + 2y \Rightarrow x = y-0 \text{ or } (y = 1 \text{ and } x^2 = 1) \Rightarrow (x, y) = (0, 0) \text{ or } (1, 1) \text{ or } (-1, 1).$$

[2 marks]

$$A = \frac{\partial^2 f}{\partial x^2} = 4 - 4y, \quad B = \frac{\partial^2 f}{\partial y \partial x} = -4x, \quad C = \frac{\partial^2 f}{\partial y^2} = 2.$$

For $(x, y) = (0, 0)$ we have $A = 4 > 0$, $B = 0$, $C = 2$ and $AC - B^2 = 8 > 0$. So $(0, 0)$ is a minimum.

For $(x, y) = (\pm 1, 1)$ $A = 0$, $B = \pm 4$, $C = 2$. So $AC - B^2 = -16 < 0$, and both these points are saddles.

[4 marks]

[2 + 2 + 4 = 8 marks]

9. For

$$f(x, y) = \frac{1}{x - y^2},$$

we have

$$\frac{\partial f}{\partial x} = \frac{-1}{(x - y^2)^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{(x - y^2)^2}.$$

So

$$f(2, 1) = 1, \quad \frac{\partial f}{\partial x}(2, 1) = -1, \quad \frac{\partial f}{\partial y}(2, 1) = 2.$$

So the linear approximation is

$$1 - (x - 2) + 2(y - 1).$$

[It would be acceptable to realise that

$$\begin{aligned} f(x, y) &= (1 + (x - 2) - 2(y - 1) - (y - 1)^2)^{-1} \\ &= (1 + (x - 2) - 2(y - 1) - (y - 1)^2)^{-1} \end{aligned}$$

and to expand out.]

[4 marks]

10.

$$\begin{aligned} \int \int_R x^2 dy dx &= \int_{-1}^1 \int_{x^2-1}^0 x^2 dy dx \\ &= \int_{-1}^1 [x^2 y]_{x^2-1}^0 dx = \int_{-1}^1 (x^2 - x^4) dx \end{aligned}$$

$$= \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_{-1}^1 = 2 \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{4}{15}.$$

[6 marks]

Section B

11. (i) We have $f^{(2k)}(x) = (-1)^k \sin x$ and $f^{(2k+1)}(x) = (-1)^k \cos x$. So $f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = (-1)^k$. So:

a) $P_4(x) = x - \frac{1}{6}x^3$ and $R_4(x) = \frac{1}{5!}x^5 \cos c$ for some c between 0 and x .

b) For any $k \geq 1$

$$P_{2k}(x) = \sum_{r=0}^{k-1} (-1)^r \frac{x^{2r+1}}{(2r+1)!}, \quad R_{2k}(x) = (-1)^k \frac{x^{2k+1}}{(2k+1)!} \cos c$$

for some c between 0 and x .

[6 marks]

Since $|\cos c| \leq 1$ for all c , and $\cos c \geq 0$ for $c \in [0, \frac{\pi}{2}]$ we have

$$0 \leq R_4(x) \leq \frac{x^5}{(5)!} \cos c \leq \frac{x^5}{120}.$$

So, since $\sin x = P_4(x) + R_4(x)$,

$$x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}.$$

If in addition $0 \leq x \leq 1$ then $x^5 \leq x^3$ and so

$$x - \frac{x^3}{6} \leq \sin x \leq x - \frac{19x^3}{120}.$$

[3 marks]

(ii) Using the Taylor series of $\sin x$ at 0:

a)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} &= \lim_{x \rightarrow 0} \frac{x - \left(x - \frac{x^3}{6} + \frac{x^5}{120} \dots \right)}{x^3} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{6} - \frac{x^5}{120} \dots}{x^3} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{6} - \frac{x^2}{120} \dots}{1} = \frac{1}{6}; \end{aligned}$$

[3 marks]

b)

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right) &= \lim_{x \rightarrow 0} \frac{\sin x - x}{x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{-\frac{x^3}{6} + \dots}{x^2 - \frac{x^4}{6} \dots} = \lim_{x \rightarrow 0} \frac{-\frac{x}{6} \dots}{1 - \frac{x^2}{6} \dots} = 0. \end{aligned}$$

[3 marks]

[6 + 3 + 3 + 3 = 15 marks.]

12. For the complementary solution in both cases, if we try $y = e^{rx}$ we need

$$r^2 + 1 = 0,$$

that is, $r = \pm i$. So $e^{\pm ix}$ is a complementary solution and in general form the complementary solution can be written $A \cos x + B \sin x$.

[3 marks]

(i). We try $y_p = Cx + D$. Then $y'_p(x) = C$ and $y''_p = 0$. So $y''_p + y_p = Cx + D = x$. So $C = 1$ and $D = 0$. So the general solution is

$$y = A \cos x + B \sin x + x$$

[3 marks]

This gives

$$y'(x) = -A \sin x + B \cos x + 1$$

So $y(0) = 1$, $y'(0) = -1$ give

$$A = 1, \quad B + 1 = -1,$$

so $B = -2$ and $A = 1$ and

$$y = \cos x - 2 \sin x + x.$$

[3 marks]

(ii) We try $y_p = Ce^{-x}$. Then $y'_p = -Ce^{-x}$ and $y''_p = Ce^{-x}$. So $y''_p + y_p = 2Ce^{-x}$. So $C = 1$. So the general solution is

$$y = A \cos x + B \sin x + e^{-x}$$

[3 marks]

This gives

$$y' = A \sin x + B \cos x - e^{-x}.$$

So putting $x = 0$ the boundary conditions give

$$A + 1 = 1, \quad B - 1 = -1 \quad \Rightarrow \quad A = B = 0.$$

So the solution is

$$e^{-x}$$

[3 marks]

[5 × 3 = 15 marks]

13. For

$$f(x, y) = x^2 + 3x^2y - y^2, \quad g(x, y) = 2x^2 + y^2,$$

$$\nabla f = (2x + 6xy)\mathbf{i} + (3x^2 - 2y)\mathbf{j}$$

$$\nabla g = 4x\mathbf{i} + 2y\mathbf{j}.$$

[2 marks]

At a stationary point of f , we have

$$2x(1 + 3y) = 3x^2 - 2y = 0.$$

So $x = 0$ or $y = -\frac{1}{3}$. If $x = 0$ then $y = 0$. If $y = -\frac{1}{3}$ then $x^2 = -\frac{2}{9}$, so there are no solutions. So the only stationary point in $g < 1$ is $(0, 0)$.

[4 marks]

At a maximum or minimum on the set where $g = 1$, we must have $\nabla f = \lambda \nabla g$. that is

$$2x + 6xy = 4\lambda x, \quad 3x^2 - 2y = 2\lambda y.$$

[1 mark]

The first equation gives $x = 0$ or $1 + 3y = 2\lambda$. If $x = 0$ then the equation $g = 1$ gives $y^2 = 1$ and $y = \pm 1$.

If $2\lambda = 1 + 3y$, then plugging into the second equation gives

$$3x^2 - 2y = y(1 + 3y)$$

So multiplying by 2 and replacing $2x^2$ by $1 - y^2$ gives

$$3 - 3y^2 - 4y = 2y + 6y^2.$$

So

$$9y^2 + 6y - 3 = 3(3y - 1)(y + 1) = 0.$$

So $y = -1$ or $y = \frac{1}{3}$ and using $g = 1$ gives

$$(x, y) = (0, -1) \text{ or } \left(\pm \frac{2}{3}, \frac{1}{3}\right).$$

So altogether the points on $g = 1$ which can be maxima or minima of f on $g \leq 1$ are

$$(0, 0) \text{ or } (0, -1) \text{ or } \left(\pm \frac{2}{3}, \frac{1}{3}\right).$$

[6 marks]

We have

$$f(0, 0) = 0, \quad f(0, -1) = -1, \quad f\left(\pm \frac{2}{3}, \frac{1}{3}\right) = \frac{7}{9}.$$

So the minimum value of f in $g \leq 1$ is -1 , realised at $(0, -1)$, and the maximum is $\frac{7}{9}$, realised at $(\pm \frac{2}{3}, \frac{1}{3})$.

[2 marks.]

[2 + 4 + 1 + 6 + 2 = 15 marks.]

14. The line $x + y = 1$ meets the y -axis $x = 0$ at $y = 1$ and the x -axis $y = 0$ at $x = 1$. So the mass is given by

$$M = \int_0^1 \int_0^{1-y} y dx dy = \int_0^1 (y - y^2) dy$$

$$= \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = \frac{1}{6}.$$

[5 marks]

Then the centre of mass is $(\bar{x}, \bar{y})$ where

$$\begin{aligned} \bar{x} &= \frac{1}{M} \int_0^1 y \int_0^{1-y} x dx dy = \frac{6}{2} \int_0^1 (y - 2y^2 + y^3) dy \\ &= 3 \left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4} \right]_0^1 = 3 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{1}{4}, \end{aligned}$$

[5 marks]

$$\begin{aligned} \bar{y} &= 6 \int_0^1 y^2 (1-y) dx dy \\ &= 6 \left[\frac{y^3}{3} - \frac{y^4}{4} \right]_0^1 = \frac{1}{2}. \end{aligned}$$

So the centre of mass is

$$\left(\frac{1}{4}, \frac{1}{2} \right).$$

[5 marks]

[5 + 5 + 5 = 15 marks]