MATH102 Solutions September 2004
Section A

1. The Taylor series of $f(x) = \ln x = \ln(1 + (x-1))$ is

$$(x-1) - \frac{(x-1)^2}{2} \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}.$$

This can also be worked out by computing all derivatives of f at $x = 1$.
[3 marks]
a) When $x = 3$ the series is not convergent and so it does not make sense to say that it is equal to $f(3)$.
[1 mark]
b) When $x = 1.5$ the series is convergent and equal to $f(1.5) = \ln(1.5) = \ln 3 - \ln 2$.
[1 mark]
No explanation is required in a) or b).
$5 = 3 + 1 + 1$ marks
2. Solving both by the Integrating factor method, we write the equations as:
(i) $\frac{dy}{dx} + \frac{2y}{x} = 0$,
(ii) $\frac{dy}{dx} + \frac{2y}{x} = 1$.
Then the integrating factor is

$$\exp\left(\int \frac{2}{x} dx\right) = \exp(2 \ln x) = \exp(\ln(x^2)) = x^2$$

Then multiplying by the integral factor we have:
(i)

$$x^2 \frac{dy}{dx} + 2xy = 0,$$
$$\frac{d}{dx}\left(x^2 y\right) = 0,$$
$$x^2 y = C \Rightarrow y = \frac{C}{x^2},$$

(ii)

$$x^2 \frac{dy}{dx} + 2xy = x^2,$$
$$\frac{d}{dx}\left(x^2 y\right) = x^2,$$
$$x^2 y = \frac{x^3}{3} + C \Rightarrow y = \frac{x}{3} + \frac{C}{x^2}.$$

3 marks for (i) 4 marks for (ii). Other methodis are possible: separation of variables for (i) and complementary and particular solutions for linear o.d.e.'s with constant coefficients

[7 marks]
3. Try $y = e^r x$. Then

$$r^2 + 5r + 6 = 0 \Rightarrow (r+2)(r+3) = 0 \Rightarrow r = -2 \text{ or } r = -3.$$

So the general solution is

$$y = Ae^{-2x} + Be^{-3x}.$$

[3 marks]
So $y' = -2Ae^{-2x} - 3Be^{-3x}$ and the initial conditions $y(0) = 1$, $y'(0) = 2$ give

$$A + B = 1, \quad -2A - 3B = 2 \rightarrow A = 5, \; B = 1 - A \Rightarrow A = 5, B = -4.$$

So

$$y = e^{-2x} - 4e^{-3x}.$$

[3 marks]
$3 + 3 = 5$ marks
4. We have

$$\lim_{x \to 0, y = 0} \frac{y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{0}{x^2} = 0,$$

[2 marks] and

$$\lim_{y \to 0, x = 0} \frac{y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{y^2}{y^2} = 1.$$

[2 marks]
$2 + 2 = 4$ marks
5.

$$\frac{\partial f}{\partial x} = 2xe^{x^2 - y^2} \sin(2xy) + 2ye^{x^2 - y^2} \cos(2xy)$$
$$\frac{\partial f}{\partial y} = -2ye^{x^2 - y^2} \sin(2xy) + 2xe^{x^2 - y^2} \cos(2xy)$$

[3 marks]

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= 2e^{x^2 - y^2} \sin(2xy) \\ &+4x^2 e^{x^2 - y^2} \sin(2xy) + 4xye^{x^2 - y^2} \cos(2xy) \\ &-4y^2 e^{x^2 - y^2} \sin(2xy) + 4xye^{x^2 - y^2} \cos(2xy) \\ &= (2 + 4(x^2 - y^2))e^{x^2 - y^2} \sin(2xy) + 8xye^{x^2 - y^2} \cos(2xy). \end{aligned}$$

Similarly

$$\begin{aligned} \frac{\partial^2 f}{\partial y^2} &= -2e^{x^2 - y^2} \sin(2xy) \\ &+4y^2 e^{x^2 - y^2} \sin(2xy) - 4xye^{x^2 - y^2} \cos(2xy) \end{aligned}$$

$$
\begin{aligned}
&-4x^2e^{x^2-y^2}\sin(2xy) - 4xye^{x^2-y^2}\cos(2xy) \\
&= (-2+4(y^2-x^2))e^{x^2-y^2}\sin(2xy) - 8xye^{x^2-y^2}\cos(2xy).
\end{aligned}
$$

which gives

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

[6 marks]
[$3+6=9$ marks]
6. Write

$$f(x,y,z) = 2x^2y + y^2z - xz^2 - 2.$$

Then

$$\nabla f(x,y,z) = (4xy - z^2)\mathbf{i} + (2x^2+2yz)\mathbf{j} + (y^2-2xz)\mathbf{k} = 3\mathbf{i}+4\mathbf{j}-\mathbf{k} \text{ at } (x,y,z) = (1,1,1).$$

[3 marks] So a normal to the surface at the point $(1,1,1)$ is given by $3\mathbf{i}-\mathbf{k}$, the normal line is given by

$$\mathbf{r} = (1+3t)\mathbf{i} + (1+4t)\mathbf{j} + (1-t)\mathbf{k}$$

and the tangent plane at this point is given by

$$3(x-1) - (z-1) = 3x + 4y - z - 6 = 0.$$

[3 marks]
[$3+3=6$ marks.]
7.

$$\frac{\partial f}{\partial x} = -6y + 2x, \quad \frac{\partial f}{\partial y} = 6y^2 - 6x.$$

[2 marks]
So at a stationary point,

$$-6y+2x = 0 = 6y^2-6x \Rightarrow x = 3y,\ 6y^2-18y = 0 \Rightarrow (y=0,\ x=0) \text{ or } (y=3,\ x=9.$$

[2 marks]

$$A = \frac{\partial^2 f}{\partial x^2} = 2,\ B = \frac{\partial^2 f}{\partial y \partial x} = 6, C = \frac{\partial^2 f}{\partial y^2} = 12y.$$

For $(x,y) = (0,0)$ we have $C = 0$, and $AC - B^2 = -36 < 0$. So $(0,0)$ is a saddle point.

For $(x,y) = (9,3)$ we have $C = 36$ and $AC - B^2 = 72 - 36 > 0$. Since also $A > 0$, $(9,3)$ is a minimum.
[4 marks]
$2+2+4=8$ marks

8. We have

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2}, \quad \frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2}.$$

So

$$f(1,0) = \ln 1 = 0, \quad \frac{\partial f}{\partial x}(1,0) = 2, \quad \frac{\partial f}{\partial y}(1,1) = 0.$$

So the linear approximation is

$$2(x-1).$$

[It would also be acceptable to realise that

$$f(x,y) = \ln(1 + 2(x-1) + (x-1)^2 + y^2)$$

and to expand out.]
[4 marks]
9.

$$\int\int_T (x-y)dxdy = \int_0^1 \int_0^{1-y} (x-y)dxdy$$

$$= \int_0^1 \left[\frac{x^2}{2} - yx\right]_0^{1-y} dy = \int_0^1 \left(\frac{(1-y)^2}{2} - y(1-y)\right) dy$$

$$= \int_0^1 \left(\frac{1}{2} - 2y + 3\frac{y^2}{2}\right) dy = \left[\frac{y}{2} - y^2 + \frac{y^3}{2}\right]_0^1 = 0.$$

[6 marks]

Section B

10. For $f(y) = (1+y)^{-1/2}$, we have

$$f'(y) = -\frac{1}{2}(1+y)^{-3/2}, \quad f''(y) = \frac{3}{4}(1+y)^{-5/2}, \quad f^{(3)}(y) = -\frac{15}{8}(1+y)^{-5/2}.$$

At $y = 0$ we have

$$f(0) = 1, \quad f'(0) = -\frac{1}{2}, \quad f''(0) = \frac{3}{4}.$$

So

$$P_2(y) = 1 - \frac{1}{2}y + \frac{3}{8}y^2$$

and

$$R_2(y) = -\frac{15}{8 \times 6}(1+c)^{-5/2}y^3$$

for some c between 0 and y.
[6 marks]

If $y = x^2$ then $y \geq 0$ and if c is betwen 0 and y we have $c \geq 0$ and

$$0 < (1+c)^{-5/2} \leq 1$$

So

$$|f(x^2) - P_2(x^2)| = |R_2(x^2)| \le \frac{5}{16}x^6.$$

[2 marks]

Now

$$\int_0^{1/3} P_2(x^2)dx = \int_0^{1/3} \left(1 - \frac{1}{2}x^2 + \frac{3}{8}x^4\right) dx$$

$$= \left[x - \frac{x^3}{6} + \frac{3}{40}x^5\right]_0^{1/3} = \frac{1}{3} - \frac{1}{162} + \frac{1}{3240} = \frac{1061}{3240} = .327469136...$$

[3 marks]

on my calculator

$$\ln(\frac{1}{3} + \frac{\sqrt{10}}{3}) = .327450150....$$

[1 mark]

Now

$$\frac{d}{dx}\ln(x + \sqrt{1+x^2}) = \frac{1 + x(1+x^2)^{-1/2}}{x + \sqrt{1+x^2}} = (1+x^2)^{-1/2}\frac{\sqrt{1+x^2+x}}{x+\sqrt{1+x^2}}$$

$$= (1+x^2)^{-1/2}.$$

So

$$\int_0^{1/3} (1+x^2)^{-1/2}dx = \left[\ln(x + \sqrt{1+x^2})\right]_0^{1/2}$$

$$= \ln(\frac{1}{3} + \frac{\sqrt{10}}{3}) - \ln 1 = \ln(\frac{1}{3} + \frac{\sqrt{10}}{3}).$$

Since $R_2(x^2)$ is small for $|x| \le \frac{1}{3}$ (in fact $\le \frac{5}{16}x^6$) we expect the difference of the integrals of $(1+x^2)^{-1/2}$ and $P_2(x^2)$ between limits 0 and $\frac{1}{2}$ to be small (in fact,

$$\le \int_0^{1/3} \frac{5x^6}{16} = \frac{5}{112 \times 2187}.$$

[3 marks]

$6 + 2 + 3 + 1 + 3 = 15$ marks

11. For the complementary solution in both cases, if we try $y = e^{rx}$ we need

$$r^2 + 1 = 0,$$

that is, $r = \pm i$. So the complementary solution can be written as $A\cos x + B\sin x$. [3 marks]

(i) We try a particular solution $y_p = Cx + D$. Then $y_p'(x) = C$ and $y_p'' = 0$. So $y_p'' + y_p = Cx + D$. Equating coefficients we get $C = 1$ and $D = 0$. So the general solution is

$$y = A\cos x + B\sin x + x.$$

[3 marks]
This gives
$$y' = -A\sin x + B\cos x + 1$$
So putting $x = 0$ the boundary conditions give
$$A = 1, \quad B + 1 = -1.$$
So
$$A = 1, \quad B = -2.$$
and the solution is
$$\cos x - 2\sin x - x.$$
[3 marks]
(ii) We try $y_p = Ce^x$. Then $y_p'' = Ce^x$. So $y_p'' + y_p = 2Ce^x$. So $C = \frac{1}{2}$. So the general solution is
$$y = A\cos x + B\sin x + \frac{1}{2}e^x.$$
[3 marks]
This gives
$$y' = -A\sin x + B\cos x + \frac{1}{2}e^x.$$
So putting $x = 0$ the boundary conditions give
$$A + \frac{1}{2} = 1, \quad B + \frac{1}{2} = -1.$$
So
$$A = \frac{1}{2}, \quad B = -\frac{3}{2}$$
and the solution is
$$\frac{1}{2}\cos x - \frac{3}{2}\sin x + \frac{1}{2}e^x.$$
[3 marks]
$5 \times 3 = 15$ marks
12. We want to minimise $f(x, y, z) = x^2 + y^2 + z^2$ (the square of the distance of (x, y, z) from $(0, 0, 0)$) subject to
$$g(x, y, z) = 4xy - 3y^2 + 2z^2 = 1.$$
[1 mark]
a)
$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$
$$\nabla XCg = 4y\mathbf{i} + (4x - 6y)\mathbf{j} + 4z\mathbf{k}.$$
At a constrained minimum we must have
$$x = 2\lambda y, \quad y = \lambda(2x - 3y), \quad z = 2\lambda z. \tag{1}$$

[3 marks]

So $2\lambda = 1$ or $z = 0$.

If $2\lambda = 1$ then $x = y$ and $-3\lambda y = 0$. So $x = y = 0$ and $2z^2 = 1$ from the equation $g(0,0,z) = 1$. So $z = \pm 1/\sqrt{2}$ and $f(0,0,\pm 1/\sqrt{2}) = 1/2$.

[3 marks]

So now let $z = 0$. We have $\lambda \neq 0$ because otherwise $x = y = 0$ which is not possible on the surface $g(x,y,z) = 1$. Subtracting x times the second equation of (1) from y times the first gives

$$2y^2 - 2x^2 + 3xy = 0.$$

This factorises as

$$(y + 2x)(2y - x) = 0.$$

[3 marks]

If $y = -2x$ then $g(x,y,0) = 1$ gives

$$-8x^2 - 12x^2 = 1,$$

which has no solutions.

If $x = 2y$ then $g(x,y,0) = 1$ gives

$$5y^2 = 1.$$

So $y = \pm 1/\sqrt{5}$ and $(x,y) = \pm(1/\sqrt{5})(2,1)$ and $f(x,y) = 4/5 + 1/5 = 1$.

[4 marks]

This is greater than the minimum distance achieved at $(0,0,\pm 1/\sqrt{2}$. So the minimum distance is $\sqrt{f(0,0,\pm 1/\sqrt{2})} = 1/\sqrt{2}$.

$1 + 3 + 3 + 3 + 4 + 1 = 15$ marks.

13. The line $2x + y = 1$ meets the y-axis $x = 0$ at $y = 1$ and the x-axis $y = 0$ at $x = \frac{1}{2}$. So the area of the triangle is given by

$$A = \int_0^{1/2} \int_0^{1-2x} dy dx = \int_0^{1/2} (1 - 2x) dx$$
$$= \left[x - x^2\right]_0^{1/2} = \frac{1}{4}.$$

[5 marks]

Then the centre of mass is $(\overline{x}, \overline{y})$ where

$$\overline{x} = 4 \int_0^{1/2} \int_0^{1-2x} x dy dx = 4 \int_0^{1/2} x(1 - 2x) dx$$
$$4\left[\frac{x^2}{2} - 2\frac{x^3}{3}\right]_0^{1/2} = 4\left(\frac{1}{8} - \frac{1}{12}\right) = \frac{1}{6}.$$

[5 marks]

$$\overline{y} = \frac{1}{A}\int_0^{1/2}\int_0^{1-2x} y dy dx = 4\int_0^{1/2}\left[\frac{y^2}{2}\right]_0^{1-2x} dx$$

$$= 4\int_0^{1/2}\left(\frac{1}{2} - 2x + 2x^2\right) dy = 4\left[\frac{x}{2} - x^2 + \frac{2x3}{3}\right]_0^{1/2}$$

$$= 4\left(\frac{1}{4} - \frac{1}{4} + \frac{1}{12}\right) = \frac{1}{3},$$

So the centre of mass is

$$\left(\frac{1}{6}, \frac{1}{3}\right).$$

[5 marks]
[5 + 5 + 5 = 15 marks]