All questions are similar to homework problems.

MATH102 Solutions May 2007
Section A

1. The Taylor series of

$$f(x) = x^{-1} = (2 + (x-2))^{-1} = 2^{-1}(1 + (x-2)/2)^{-1}$$

is

$$\frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{2^{n+1}}.$$

This can also be worked out by computing all derivatives of $f$ at $x = 2$.
[3 marks]
a) When $x = 1$ the series is convergent.
[1 mark]
b) When $x = 4$ the series is not convergent.
[1 mark]
No explanation is required in a) or b).
$5 = 3 + 1 + 1$ marks

2(i) Separating the variables, we have

$$\int e^y dy = \int x dx,$$

$$e^y = \frac{x^2}{2} + C.$$

Putting $x = 1$ and $y = 0$ gives

$$1 = \frac{1}{2} + C$$

or $C = \frac{1}{2}$. So we obtain

$$y = \ln\left(\frac{x^2 + 1}{2}\right).$$

It is acceptable to leave the answer in the form $e^y = (x^2 + 1)/2$.
2(ii) In standard form, the equation becomes

$$\frac{dy}{dx} + \frac{2}{x} y = 1.$$

Using the integrating factor method, the integrating factor is

$$\exp\left(\int (2/x) dx\right) = x^2.$$

So the equation becomes
$$\frac{d}{dx}(yx^2) = x^2.$$
Integrating gives
$$yx^2 = \int x^2 dx = \frac{x^3}{3} + C.$$
So the general solution is
$$y = \frac{x}{3} + Cx^{-2}.$$
Putting $y(1) = 0$ gives $C = -\frac{1}{3}$ and
$$y = \frac{x}{3} - \frac{1}{3}x^{-2}.$$

3 marks for (i) 5 marks for (ii).
[8 marks]

3. Try $y = e^{rx}$. Then
$$r^2 - 4r + 3 = 0 \Rightarrow (r-3)(r-1) = 0 \Rightarrow r = 3 \text{ or } r = 1.$$

So the general solution is
$$y = Ae^{3x} + Be^x.$$

[2 marks]
So $y' = 3Ae^{3x} + Be^x$ and the initial conditions $y(0) = 2$, $y'(0) = 1$ give
$$A + B = 2, \;\; 3A + B = 1 \Rightarrow 2A = -1, \; B = 2 - A \Rightarrow A = -\frac{1}{2}, \; B = \frac{5}{2}.$$
So
$$y = -\frac{1}{2}e^{3x} + \frac{5}{2}e^x.$$

[3 marks]
[$2 + 3 = 5$ marks]
4. We have
$$\lim_{(x,y)\to(0,0), y=0} \frac{xy}{x^2 + xy + y^2} = \lim_{x\to 0} \frac{0}{x^2} = 0,$$
$$\lim_{(x,y)\to(0,0), y=x} \frac{xy}{x^2 + xy + y^2} = \lim_{x\to 0} \frac{x^2}{3x^2} = \frac{1}{3}.$$
So the limits along two different lines as $(x,y) \to (0,0)$ are different, and the overall limit does not exist.
[4 marks]

5.
$$\frac{\partial f}{\partial x} = 4x^3 - 12xy^2,$$

$$\frac{\partial f}{\partial y} = 4y^3 - 12x^2 y,$$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 12y^2,$$

$$\frac{\partial^2 f}{\partial y \partial x} - 24xy,$$

$$\frac{\partial^2 f}{\partial x \partial y} = -24xy$$

so that these last two are equal, and

$$\frac{\partial^2 f}{\partial y^2} = 12y^2 - 12x^2.$$

So we also have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

as required.[5 marks]

6. We have

$$\frac{\partial f}{\partial x} = 2xy + yz\cos(xyz),$$

$$\frac{\partial f}{\partial y} = x^2 + xz\cos(xyz),$$

$$\frac{\partial f}{\partial z} = xy\cos(xyz)$$

[3 marks]

By the Chain Rule,

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

So

$$\begin{aligned}\frac{dF}{dt}(0) &= \frac{\partial f}{\partial x}(2,-1,0) + \frac{\partial f}{\partial y}(2,-1,0) - \frac{\partial f}{\partial z}(2,-1,0)\\ &= -4 + 4 - (-2) = 2.\end{aligned}$$

[2 marks]
[3 + 2 = 5 marks]

7. For

$$f(x,y,z) = \frac{xyz}{x^2+y^2+z^2}$$

we have

$$\nabla f(x,y,z) = \left(\frac{yz}{x^2+y^2+z^2} - \frac{2x^2yz}{(x^2+y^2+z^2)^2}\right)\mathbf{i}$$

$$+\left(\frac{xz}{x^2+y^2+z^2}-\frac{2xy^2z}{(x^2+y^2+z^2)^2}\right)\mathbf{j}+\left(\frac{xy}{x^2+y^2+z^2}-\frac{2xyz^2}{(x^2+y^2+z^2)^2}\right)\mathbf{k}.$$

So

$$\nabla f(1,1,1)=\left(\frac{1}{3}-\frac{2}{9}\right)(\mathbf{i}+\mathbf{j}+\mathbf{k})=\frac{1}{9}(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

3 marks

The derivative of $f$ in the direction $\mathbf{i}-2\mathbf{j}-2\mathbf{k}$ is

$$\frac{\nabla f(1,1,1).(\mathbf{i}-2\mathbf{j}-2\mathbf{k})}{\sqrt{1+(-2)^2+(-2)^2}}=\frac{1}{9}\times\frac{-3}{3}=-\frac{1}{9}.$$

[2 marks]
[$3+2=5$ marks.]

8. For

$$f(x,y)=x^2y-2xy+y^2-15y,$$

we have

$$\frac{\partial f}{\partial x}=2xy-2y \quad \frac{\partial f}{\partial y}=x^2-2x+2y-15.$$

[2 marks]

So at a stationary point,

$$2y(x-1)=0=x^2-2x+2y-15$$

$$\Leftrightarrow (x,y)=(1,8) \text{ or } (-3,0) \text{ or } (5,0).$$

[2 marks]

$$A=\frac{\partial^2 f}{\partial x^2}=2y,\ B=\frac{\partial^2 f}{\partial y\partial x}=2x-2, C=\frac{\partial^2 f}{\partial y^2}=2.$$

For $(x,y)=(1,8)$, $A=16$, $B=0$ and $C=2$. So $AC-B^2=32>0$ $A>0$ and $(1,8)$ is a local minimum.

For $(x,y)=(-3,0)$, we have $A=0$, $B=-8$, $C=2$. So $AC-B^2<0$, and $(-3,0)$ is a saddle.

For $(x,y)=(5,0)$, we have $A=0$, $B=8$, $C=2$. So $AC-B^2<0$, and $(5,0)$ is again a saddle.

[4 marks]
[$2+2+4=8$ marks]

9. For

$$f(x,y)=\frac{1}{x^2-y^2},$$

we have

$$\frac{\partial f}{\partial x}=\frac{-2x}{(x^2-y^2)^2},\quad \frac{\partial f}{\partial y}=\frac{2y}{(x^2-y^2)^2}.$$

So

$$f(2,1) = \frac{1}{3}, \quad \frac{\partial f}{\partial x}(2,1) = -\frac{4}{9}, \quad \frac{\partial f}{\partial y}(2,1) = \frac{2}{9}.$$

So the linear approximation is

$$\frac{1}{3} - \frac{4}{9}(x-2) + \frac{2}{9}(y-1).$$

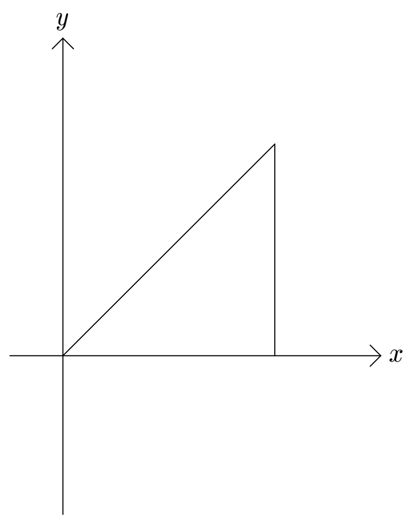
[It would be acceptable to realise that

$$f(x,y) = (3 + 4(x-2) + (x-2)^2 - 2(y-1)(y-1)^2)^{-1}$$

$$= \frac{1}{3}\left(1 + \frac{4}{3}(x-1) - \frac{2}{3}(y-1) + \frac{1}{3}(x-2)^2 - \frac{1}{3}(y-1)^2\right)^{-1}$$

and to expand out.]
[4 marks]

10. The domain of integration is the triangle as shown




This integral can be written as $\int_0^1 \int_0^x dydx$ or $\int_0^1 \int_y^1 dxdy$. So we have

$$\int_0^1 \int_1^y e^{y/x} dxdy = \int_0^1 \int_0^x e^{y/x} dxdy$$

$$= \int_0^1 \left[xe^{y/x}\right]_{y=0}^{y=x} dx = \int_0^1 x(e-1)dx$$

$$= \left[(e-1)\frac{x^2}{2}\right]_0^1 = \frac{e-1}{2}.$$

[6 marks]

Section B

11. (i) The Taylor series of $f$ at 0 is

$$1 - y + y^2 \cdots = \sum_{n=0}^{\infty} (-1)^n y^n.$$

[3 marks]

Putting $y = x^2$, the Taylor series of $g$ at 0 is

$$\sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

[2 marks]

Integrating, the Taylor series of $h(x) = \tan^{-1}(x)$ at 0 is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

[2 marks]

(ii) We have

$$f^{(n+1)}(y) = (-1)^{n+1}(n+1)!(1+y)^{-n-2}.$$

Now

$$R_n(y,0) = \frac{f^{(n+1)}(c)}{(n+1)!} y^{n+1} = (-1)^{n+1}(1+c)^{-n-2} y^{n+1}$$

for some $c$ between 0 and $y$. Since $c \geq 0$, $|(1+c)^{-n-2}| \leq 1$. So

$$|R_n(y,0) \leq |(-1)^{n+1} y^{n+1}| \leq y^{n+1}.$$

[3 marks]

So

$$\begin{aligned} |\int_0^x R_n(t^2,0)dt| &\leq \int_0^x |R_n(t^2,0)|dt \\ &\leq \int_0^x t^{2n+2} dt = \frac{x^{2n+3}}{2n+3}. \end{aligned}$$

[2 marks]

(iii)

$$h(1) = \tan^{-1}(1) = \frac{\pi}{4}.$$

$$\begin{aligned} P_{22}(1,0) &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \frac{1}{21} \\ &= \frac{2}{3} + \frac{2}{35} + \frac{2}{99} + \frac{2}{195} + \frac{2}{323} + \frac{1}{21} \\ &= 0.808078952..... \end{aligned}$$

Meanwhile

$$\frac{\pi}{4} = 0.785398163...$$

So the difference is $< 0.0247$. This is $< 1/23 = 0.0434..$ as required.
[3 marks]
$3+2+2+3+2+3 = 15$ marks.

12. For the complementary solution in both cases, if we try $y = e^{rx}$ we need

$$r^2 - 4r - 5 = (r-5)(r+1) = 0,$$

that is, $r = 5$ or $-1$. So the complementary solution is $Ae^{5x} + Be^{-x}$.
[3 marks]
(i) We try $y_p = Ce^x$. Then $y_p' = Ce^x = y_p''$. So $y_p'' - 4y_p' - 5y_p = -8C$. So $C = -\frac{1}{2}$ So the general solution is

$$y = Ae^{5x} + Be^{-x} - \frac{1}{2}e^x.$$

[2 marks]
This gives

$$y' = 5Ae^{5x} - Be^{-x} - \frac{1}{2}e^x.$$

So putting $x = 0$, the boundary conditions give

$$A+B-\frac{1}{2} = 1, \;\; 5A-B-\frac{1}{2} = -1 \;\; \Rightarrow \;\; 6A = 1, \;\; B = \frac{3}{2} - A \Rightarrow A = \frac{1}{6}, \;\; B = \frac{4}{3}.$$

So the solution is

$$y = \frac{1}{6}e^{5x} + \frac{4}{3}e^{-x} - \frac{1}{2}e^x.$$

[3 marks]
(ii) We try $y_p = Cx^2 + Dx + E$. Then $y_p'(x) = 2Cx + D$ and $y_p'' = 2C$. So

$$y_p'' - 4y_p' - 5y_p = (2C - 4D - 5E) + x(-8C - 5D) - 5Cx^2 = -5x^2 + 2x + 5.$$

Comparing coefficients, we obtain

$$-5C = -5, \;\; -8C - 5D = 2, \;\; 2C - 4D - 5E = 5.$$

So

$$C = 1, \;\; D = -2, \;\; 10 - 5E = 5 \Rightarrow E = 1$$

So the general solution is

$$Ae^{5x} + Be^{-x} + x^2 - 2x + 1.$$

[4 marks] This gives

$$y'(x) = 5Ae^{5x} - Be^{-x} + 2x - 2.$$

So putting $x = 0$, the boundary conditions give

$$A + B + 1 = 1, \quad 5A - B - 2 = -1 \Rightarrow A = -B, \quad 6A = 1 \Rightarrow A = \frac{1}{6}, \quad B = -\frac{1}{6}.$$

So

$$y = \frac{1}{6}e^{5x} - \frac{1}{6}e^{-x} + x^2 - 2x + 1.$$

[3 marks]
[$3 + 2 + 3 + 4 + 3 = 15$ marks]

13a)

We have

$$\nabla f = (y+1)\mathbf{i} + x\mathbf{j}$$
$$\nabla g = 6x\mathbf{i} + 2y\mathbf{j}.$$

[2 marks]

At a stationary point of $f$ we have

$$y + 1 = x = 0 \Rightarrow (x, y) = (0, -1).$$

This is in the set where $g(x,y) < 3$. (The point is easily seen to be a saddle and so cannot be a maximum of minimum of $f$ on the set where $g \leq 3$, but we shall not use this.)
[2 marks]

At a stationary point of $f$ on $g = 3$, we have $\nabla f = \lambda \nabla g$, that is,

$$y + 1 = 6x\lambda, \quad x = 2y\lambda \Rightarrow y(y+1) - 3x^2 = 0.$$

On $g = 3$, we have $3x^2 = 3 - y^2$, so

$$2y^2 + y - 3 = (2y+3)(y-1) = 0.$$

So $y = 1$ or $y = -\frac{3}{2}$. So the stationary points of $f$ restricted to $g = 3$ are

$$(\pm\sqrt{2/3}, 1), \quad \left(\pm\frac{1}{2}, -\frac{3}{2}\right).$$

[6 marks]

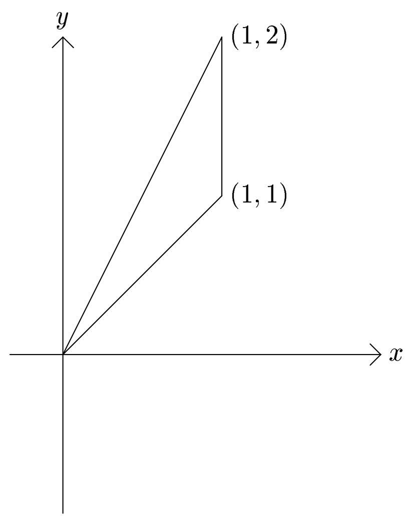
Now we check the values of $f$ at all these points. We have

$$f(0,-1) = 0, \quad f(\sqrt{2/3}, 1) = 2\sqrt{2/3}, \quad f(-\sqrt{2/3}, 1) = -2\sqrt{2/3},$$
$$f\left(\frac{1}{2}, -\frac{3}{2}\right) = -\frac{1}{4}, \quad f\left(-\frac{1}{2}, -\frac{3}{2}\right) = \frac{1}{4}.$$

So the minimum value is $-2\sqrt{2/3}$ achieved as $(-\sqrt{2/3}, 1)$ and the maximum value is $2\sqrt{2/3}$, achieved at $(\sqrt{2/3}, 1)$.

[5 marks]
[$2 + 2 + 6 + 5 = 15$ marks.]

14a).The region $R$ is as shown.




The weight $W$ is

$$\int_0^1 \int_x^{2x} x dy dx = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

[5 marks]
14b) Then

$$\overline{x} = \frac{1}{W} \int_0^1 \int_x^{2x} x^2 dy dx$$
$$= 3 \int_0^1 x^3 dx = 3 \left[\frac{x^4}{4}\right]_0^1 = \frac{3}{4}.$$

[5 marks]

$$\overline{y} = \frac{1}{W} \int_0^1 \int_x^{2x} xy dy dx$$
$$= 3 \int_0^1 x \left[\frac{y^2}{2}\right]_x^{2x} dx = \frac{9}{2} \int_0^1 x^3 dx$$
$$= \frac{9}{2} \left[\frac{x^4}{4}\right]_0^1 = \frac{9}{8}.$$

So

$$(\overline{x}, \overline{y}) = \left(\frac{3}{4}, \frac{9}{8}\right).$$

[5 marks]

[$3 \times 5 = 15$ marks.]