

All questions are similar to homework problems.

MATH102 Solutions May 2006

Section A

1. The Taylor series of $f(x) = x^{-2} = (1 + (x - 1))^{-2}$ is

$$1 - 2(x - 1) + \frac{2 \cdot 3}{2!}(x - 1)^2 - \frac{2 \cdot 3 \cdot 4}{3!}(x - 1)^3 \cdots = \sum_{n=0}^{\infty} (-1)^n (n + 1)(x - 1)^n.$$

This can also be worked out by computing all derivatives of f at $x = 1$.

[3 marks]

- a) When $x = 0.5$ the series is convergent and equal to $f(0.5) = 4$.

[1 mark]

- b) When $x = 2$ the series is not convergent to $f(2) = \frac{1}{4}$. In fact, the series is not convergent.

[1 mark]

No explanation is required in a) or b).

5 = 3 + 1 + 1 marks

- 2(i) Separating the variables, we have

$$\int y dy = - \int \sin x dx,$$

$$\frac{y^2}{2} = \cos x + C,$$

or

$$y = \pm \sqrt{2C + 2 \cos x}.$$

Putting $y(0) = 1$ gives $2C = -1$ and $y = +\sqrt{2 \cos x - 1}$.

- 2(ii) Using the integrating factor method, and the integrating factor is

$$\exp \left(\int dx \right) = e^{-x}.$$

So the equation becomes

$$\frac{d}{dx}(ye^{-x}) = e^x.$$

Integrating gives

$$ye^{-x} = e^x + C.$$

So the general solution is

$$y = e^{2x} + Ce^x.$$

Putting $y(0) = 2$ gives $C = 1$ and $y = e^{2x} + e^x$.

3 marks for (i) 4 marks for (ii).

[7 marks]

3. Try $y = e^r x$. Then

$$r^2 + 2r - 15 = 0 \Rightarrow (r - 3)(r + 5) = 0 \Rightarrow r = -5 \text{ or } r = 3.$$

So the general solution is

$$y = Ae^{3x} + Be^{-5x}.$$

[2 marks]

So $y' = 3Ae^{3x} - 5Be^{-5x}$ and the initial conditions $y(0) = 2$, $y'(0) = -1$ give

$$A + B = 2, \quad 3A - 5B = -1 \rightarrow 8B = 7, \quad A = 2 - B \Rightarrow B = \frac{7}{8}, \quad A = \frac{9}{8}.$$

So

$$y = \frac{9}{8}e^{3x} + \frac{7}{8}e^{-5x}.$$

[3 marks]

[2 + 3 = 5 marks]

4. We have

$$\lim_{(x,y) \rightarrow (0,0), y=0} \frac{x^2 y^2}{x^4 + y^4} = \lim_{x \rightarrow 0} \frac{0}{x^4} = 0,$$
$$\lim_{(x,y) \rightarrow (0,0), y=x} \frac{x^2 y^2}{x^4 + y^4} = \lim_{x \rightarrow 0} \frac{x^4}{2x^4} = \frac{1}{2}.$$

So the limits along two different lines as $(x, y) \rightarrow (0, 0)$ are different, and the overall limit does not exist.

[4 marks]

5.

$$\frac{\partial f}{\partial x} = 2xy \cos(x^2 y), \quad \frac{\partial f}{\partial y} = x^2 \cos(x^2 y),$$

$$\frac{\partial^2 f}{\partial x^2} = 2y \cos(x^2 y) - 4x^2 y^2 \sin(x^2 y),$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2x \cos(x^2 y) - 2x^3 y \sin(x^2 y),$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x \cos(x^2 y) - 2x^3 y \sin(x^2 y),$$

so that these last two are equal, and

$$\frac{\partial^2 f}{\partial y^2} = -x^4 \sin(x^2 y).$$

[5 marks]

6. We have

$$\frac{\partial u}{\partial x} = 2, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = -1, \quad \frac{\partial v}{\partial y} = 2.$$

By the Chain Rule,

$$\begin{aligned} \frac{\partial g}{\partial x}(x, y) &= \frac{\partial f}{\partial u}(u, v) \frac{\partial u}{\partial x}(x, y) - \frac{\partial f}{\partial v}(u, v) \frac{\partial v}{\partial x}(x, y) \\ &= 2 \frac{\partial f}{\partial u}(u, v) - \frac{\partial f}{\partial v}(u, v). \end{aligned}$$

Similarly,

$$\frac{\partial g}{\partial y}(x, y) = \frac{\partial f}{\partial u}(u, v) + 2 \frac{\partial f}{\partial v}(u, v).$$

Similarly,

$$\begin{aligned} \frac{\partial^2 g}{\partial x^2}(x, y) &= 2 \left(2 \frac{\partial^2 f}{\partial u^2}(u, v) - \frac{\partial^2 f}{\partial u \partial v}(u, v) \right) - \left(2 \frac{\partial^2 f}{\partial v \partial u}(u, v) - \frac{\partial^2 f}{\partial v^2}(u, v) \right) \\ &= 4 \frac{\partial^2 f}{\partial u^2}(u, v) + \frac{\partial^2 f}{\partial v^2}(u, v) - 4 \frac{\partial^2 f}{\partial u \partial v}(u, v). \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial^2 g}{\partial y^2}(x, y) &= \left(\frac{\partial^2 f}{\partial u^2}(u, v) + 2 \frac{\partial^2 f}{\partial u \partial v}(u, v) \right) + 2 \left(\frac{\partial^2 f}{\partial v \partial u}(u, v) + 2 \frac{\partial^2 f}{\partial v^2}(u, v) \right) \\ &= \frac{\partial^2 f}{\partial u^2}(u, v) + 4 \frac{\partial^2 f}{\partial v^2}(u, v) + 4 \frac{\partial^2 f}{\partial u \partial v}(u, v). \end{aligned}$$

Adding, we obtain

$$\frac{\partial^2 g}{\partial x^2}(x, y) + \frac{\partial^2 g}{\partial y^2}(x, y) = 5 \left(\frac{\partial^2 f}{\partial u^2}(u, v) + \frac{\partial^2 f}{\partial v^2}(u, v) \right).$$

[6 marks]

7. For

$$f(x, y, z) = \frac{1}{xy} + \frac{1}{yz} + \frac{1}{xz}$$

we have

$$\nabla f(x, y, z) = (-x^{-2}(y^{-1} + z^{-1})\mathbf{i} + (-y^{-2}(x^{-1} + z^{-1})\mathbf{j} + (-z^{-2}(x^{-1} + y^{-1})\mathbf{k}.$$

So

$$\nabla f(1, 1, 1) = -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

3 marks

The tangent plane at $(1, 1, 1)$ is

$$\nabla f(1, 1, 1) \cdot ((x - 1)\mathbf{i} + (y - 1)\mathbf{j} + (z - 1)\mathbf{k}) = 0,$$

or

$$-2(x-1) - 2(y-1) - 2(z-1) = 0,$$

or

$$x + y + z - 3 = 0.$$

[2 marks]

[3 + 2 = 5 marks.]

8. For

$$f(x, y) = y^2x + 2yx + 2x^2 - 3x,$$

we have

$$\frac{\partial f}{\partial x} = y^2 + 2y + 4x - 3, \quad \frac{\partial f}{\partial y} = x(2y + 2).$$

[2 marks]

So at a stationary point,

$$x(2y + 2) = 0 = y^2 + 2y + 4x - 3 \Leftrightarrow x = 0 = (y + 3)(y - 1) \text{ or } y + 1 = 0 = 4x - 4$$

$$\Leftrightarrow (x, y) = (0, 1) \text{ or } (0, -3) \text{ or } (1, -1).$$

[2 marks]

$$A = \frac{\partial^2 f}{\partial x^2} = 4, \quad B = \frac{\partial^2 f}{\partial y \partial x} = 2y + 2, \quad C = \frac{\partial^2 f}{\partial y^2} = 2x.$$

For $(x, y) = (0, 1)$ or $(0, -3)$ we have $C = 0$ and $B \neq 0$. So $AC - B^2 < 0$ and these points are saddle points.

For $(x, y) = (1, -1)$, we have $A = 4$, $B = 0$, $C = 2$. So $A > 0$, $AC - B^2 > 0$ and $(1, -1)$ is a minimum.

[4 marks]

[2 + 2 + 4 = 8 marks]

9. For

$$f(x, y) = \frac{1}{x^2 + y^2},$$

we have

$$\frac{\partial f}{\partial x} = \frac{-2x}{(x^2 + y^2)^2}, \quad \frac{\partial f}{\partial y} = \frac{-2y}{(x^2 + y^2)^2}.$$

So

$$f(1, 1) = \frac{1}{2}, \quad \frac{\partial f}{\partial x}(1, 1) = -\frac{1}{2}, \quad \frac{\partial f}{\partial y}(1, 1) = -\frac{1}{2}.$$

So the linear approximation is

$$\frac{1}{2} - \frac{1}{2}(x - 1) - \frac{1}{2}(y - 1).$$

[It would be acceptable to realise that

$$\begin{aligned} f(x, y) &= (2 + 2(x - 1) + (x - 1)^2 + 2(y - 1) + (y - 1)^2)^{-1} \\ &= \frac{1}{2}(1 + (x - 1) + (y - 1) + \frac{1}{2}(x - 1)^2 + \frac{1}{2}(y - 1)^2)^{-1} \end{aligned}$$

and to expand out.]

[4 marks]

10. In polar coordinates (r, θ) , D is the set where $r \leq 1$ and $0 \leq \theta \leq 2\pi$ (by choice of argument). Also, $x^2 + y^2 = r^2$ and $dxdy = r dr d\theta$. So

$$\begin{aligned} \int \int_D e^{x^2+y^2} dxdy &= \int_0^{2\pi} \int_0^1 r e^{r^2} dr = \int_0^{2\pi} \left[\frac{1}{2} e^{r^2} \right]_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi} \frac{1}{2} (e - 1) d\theta = \pi(e - 1). \end{aligned}$$

[6 marks]

Section B

11. (i) a) $f'(z) = \frac{1}{2}(4 + z)^{-1/2}$, $f''(z) = -\frac{1}{4}(4 + z)^{-3/2}$. So $f(0) = 2$, $f'(0) = \frac{1}{4}$ and

$$P_1(z, 0) = 2 + \frac{1}{4}z, \quad R_1(z, 0) = -\frac{1}{8}(4 + c)^{-3/2}z^2$$

for some c between 0 and z .

[3 marks]

If $|z| \leq 2$ and c is between 0 and z then $(4 + c)^{-3/2} \leq 2^{-3/2}$ and

$$|R_1(z, 0)| \leq \frac{1}{8 \cdot 2\sqrt{2}} \cdot 2^2 = \frac{1}{4\sqrt{2}}.$$

[3 marks]

(i) b) $g'(x) = -\sin x$, $g''(x) = -\cos x$, $g^{(3)}(x) = \sin x$, $g^{(4)}(x) = \cos x$. So $g(0) = 1$, $g'(0) = 0$, $g''(0) = -1$, $g^{(3)}(0) = 0$, and

$$P_3(x, 0) = 1 - \frac{x^2}{2}, \quad R_3(x, 0) = \frac{\cos c}{4!} x^4$$

for some c between 0 and x .

[4 marks]

Since $|\cos c| \leq 1$ we have

$$|R_3(x, 0)| \leq \frac{x^4}{24}.$$

[1 mark]

(ii) $y^2 = 8 + 2(\cos x - 1)$. Then $y = \sqrt{2}f(\cos x - 1)$. So for $P_1(z, 0)$ and $R_1(z, 0)$ as in (i)a),

$$\begin{aligned} y &= \sqrt{2}P_1(\cos x - 1, 0) + \sqrt{2}R_1(c, 0) \\ &= \sqrt{2}\left(\frac{7}{4} + \frac{1}{4}\cos x\right) + R_1(c, 0) \end{aligned}$$

for some c between 0 and $\cos x - 1$. Since $-2 \leq \cos x - 1 \leq 0$ for all x , we have $-2 \leq c \leq 0$, and by (i)a) $\sqrt{2}|R_1(\cos x - 1, 0)| \leq \frac{1}{4}$.
[4 marks.]

12. For the complementary solution in both cases, if we try $y = e^{rx}$ we need

$$r^2 + 4r + 3 = (r + 1)(r + 3) = 0,$$

that is, $r = -1$ or -3 . So the complementary solution is $Ae^{-x} + Be^{-3x}$.
[3 marks]

(i) We try $y_p = Cx + D$. Then $y'_p = C$ and $y''_p = 0$. So $y''_p + 4y'_p + 3y_p = 4C + 3Cx + 3D = 3x + 1$. So $C = 1$ and $3D = 1 - 4C$, that is, $D = -1$. So the general solution is

$$y = Ae^{-x} + Be^{-3x} + x - 1.$$

[3 marks]

This gives

$$y' = -Ae^{-x} - 3Be^{-3x} + 1.$$

So putting $x = 0$, the boundary conditions give

$$A + B - 1 = 1, \quad -A - 3B + 1 = 2 \Rightarrow -2B = 3, \quad A = 2 - B \Rightarrow B = -\frac{3}{2}, \quad A = \frac{7}{2}.$$

So the solution is

$$y = \frac{7}{2}e^{-x} - \frac{3}{2}e^{-3x} + x - 1.$$

[3 marks]

(ii) We try $y_p = C \sin x + D \cos x$. Then $y'_p(x) = C \cos x - D \sin x$ and $y''_p = -C \sin x - D \cos x$. So

$$y''_p + 4y'_p + 3y_p = (2C - 4D) \sin x + (4C + 2D) \cos x = 5 \sin x.$$

So $D = -2C$ and $10C = 5$ So the general solution is

$$y = Ae^{-x} + Be^{-3x} + \frac{1}{2} \sin x - \cos x.$$

[3 marks] This gives

$$y'(x) = -Ae^{-x} - 3Be^{-3x} + \frac{1}{2} \cos x + \sin x$$

So putting $x = 0$, the boundary conditions give

$$A + B - 1 = 1, \quad -A - 3B + \frac{1}{2} = 2 \Rightarrow -2B = \frac{7}{2}, A = 2 - B \Rightarrow B = -\frac{7}{4}, A = \frac{15}{4}.$$

So

$$y = \frac{15}{4}e^{-x} - \frac{7}{4}e^{-3x} + \frac{1}{2}\sin x - \cos x.$$

[3 marks]

[5 × 3 = 15 marks]

13a) The area is $A(x, y) = 4xy$ if the vertices of the rectangle are at $(\pm x, \pm y)$ with $x \geq 0, y \geq 0$.

[1 mark]

We have

$$\nabla A = 4y\mathbf{i} + 4x\mathbf{j}$$

$$\nabla g = 6x\mathbf{i} + 10y\mathbf{j}.$$

[2 marks]

At a stationary point of A on $g = 10$, we have $\nabla A = \lambda g$, that is,

$$4y = 6x\lambda, \quad 4x = 10y\lambda \Rightarrow (6x^2 - 10y^2)\lambda = 0.$$

If $\lambda = 0$ then $x = y = 0$, which is inconsistent with $g = 10$. So $6x^2 - 10y^2 = 0$ and $x = \pm\sqrt{5/3}y$. The condition $g = 10$ then gives $10y^2 = 10$ and $y = \pm 1$. So the maximum area is $4\sqrt{5/3}$.

[5 marks]

b) We have

$$\nabla f = 2x\mathbf{i} - 2y\mathbf{j}.$$

[1 mark]

The only stationary point of f is $2x = -2y = 0$, that is, $(x, y) = (0, 0)$, which is inside the ellipse and $f(0, 0) = 0$. [One can note that this is a saddle point and therefore not a local maximum or minimum but this turns out to be unnecessary.]

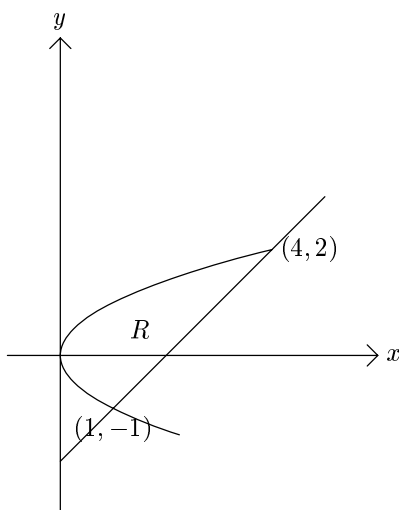
2 marks.

For a stationary point on the ellipse, we have $\nabla A = \lambda g$, that is $2x = 6x\lambda$ and $-2y = 10y\lambda$. So either $x = 0$ and $\lambda = -\frac{1}{5}$; or $\lambda = 3$ and $y = 0$. We cannot have $x = y = 0$ since $g = 10$. If $x = 0$ then the condition $g = 10$ gives $y^2 = 2$ and $f = -2$. If $y = 0$ then the condition $g = 10$ gives $3x^2 = 10$ and $f = \frac{10}{3}$. So the maximum and minimum values of f are $\frac{10}{3}$ and -2 .

[1 + 2 + 5 + 1 + 2 + 4 = 15 marks.]

14a). The line $y = x - 2$ meets the parabola $x = y^2$ when $y^2 - y - 2 = 0$, that is, $(y + 1)(y - 2) = 0$. When $y = -1$ then $x = 1$ and when $y = 2$ $x = 4$ The

parabola is to the left of the line. The region R is as shown.



[3 marks]

The area A is

$$\begin{aligned} \int_{-1}^2 \int_{y^2}^{y+2} dx dy &= \int_{-1}^2 (y + 2 - y^2) dy \\ &= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2 = 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = \frac{9}{2} \end{aligned}$$

[4 marks]

14b) Then

$$\begin{aligned} \bar{x} &= \frac{1}{A} \int_{-1}^2 \int_{y^2}^{y+2} x dx dy \\ &= \frac{2}{9} \int_{-1}^2 \left[\frac{x^2}{2} \right]_{y^2}^{y+2} dy = \frac{1}{9} \int_{-1}^2 (y^2 + 4y + 4 - y^4) dy \\ &= \frac{1}{9} \left[\frac{y^3}{3} + 2y^2 + 4y - \frac{y^5}{5} \right]_{-1}^2 = \frac{1}{9} \left(\frac{8}{3} + 8 + 8 - \frac{32}{5} + \frac{1}{3} - 2 + 4 - \frac{1}{5} \right) \\ &= \frac{1}{9} \cdot \frac{72}{5} = \frac{8}{5} \end{aligned}$$

[4 marks]

$$\begin{aligned} \bar{y} &= \frac{1}{A} \int_{-1}^2 \int_{y^2}^{y+2} y dx dy \\ &= \frac{2}{9} \int_{-1}^2 (y^2 + 2y - y^3) dy = \frac{2}{9} \left[\frac{y^3}{3} + y^2 - \frac{y^4}{4} \right]_{-1}^2 \end{aligned}$$

$$= \frac{2}{9} \left(\frac{8}{3} + 4 - 4 + \frac{1}{3} - 1 + \frac{1}{4} \right) = \frac{2}{9} \cdot \frac{9}{4} = \frac{1}{2}.$$

So

$$(\overline{x}, \overline{y}) = \left(\frac{8}{5}, \frac{1}{2} \right).$$

[4 marks]

[3 + 4 + 4 + 4 = 15 marks.]