SECTION A

1. Write down the Taylor series about x = 1 for the function

$$f(x) = x^{-2}.$$

State whether this Taylor series converges to f(x) for:

- a) x = 0.5,
- b) x = 2.

[5 marks]

2. Find the solutions of the following differential equations:

- (i) $y \frac{dy}{dx} + \sin x = 0$ with y(0) = 1,
- (ii) $\frac{dy}{dx} y = e^{2x}$ with y(0) = 2.

[7 marks]

3. Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 15y = 0$$

with the initial conditions y(0) = 2, y'(0) = -1.

[5 marks]

4. Show, by taking limits along two different paths to the origin (0,0), that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4}$$

does not exist.

[4 marks]

5. Work out all first and second partial derivatives of

$$f(x,y) = \sin(x^2 y),$$

and verify that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

[5 marks]

6. Suppose that f is a function with continuous first and second partial derivatives, and that g(x,y) = f(u,v), where u(x,y) = 2x + y and v(x,y) = -x + 2y. Use the Chain Rule to verify that

$$\frac{\partial^2 g}{\partial x^2}(x,y) + \frac{\partial^2 g}{\partial y^2}(x,y) = 5\left(\frac{\partial^2 f}{\partial u^2}(u,v) + \frac{\partial^2 f}{\partial v^2}(u,v)\right).$$

[6 marks]

7. Find the gradient vector $\nabla f(1,1,1)$, where

$$f(x, y, z) = \frac{1}{xy} + \frac{1}{yz} + \frac{1}{xz}.$$

Find the tangent plane at (1,1,1) of the surface f(x,y,z)=3.

[5 marks]

8. Locate and classify all stationary points of the function

$$f(x,y) = y^2x + 2yx + 2x^2 - 3x.$$

[8 marks]

9. Find the linear approximation near (x, y) = (1, 1) to the function

$$f(x,y) = \frac{1}{x^2 + y^2}.$$

[4 marks]

10. By using polar coordinates, evaluate the double integral

$$\int \int_D e^{x^2 + y^2} dx dy,$$

where D is the unit disc $\{(x,y): x^2 + y^2 \le 1\}$.

[6 marks]

SECTION B

11.

(i) a) Find the degree 1 Taylor polynomial $P_1(z,0)$ and the remainder term $R_1(z,0)$ for $f(z)=\sqrt{4+z}$. Show that if $|z|\leq 2$ then

$$|R_1(z,0)| \le \frac{1}{4\sqrt{2}}.$$

b) Find the Taylor polynomial $P_3(x,0)$ and the remainder term $R_3(x,0)$ for $g(x)=\cos x$. Show that

$$|R_3(x,0)| \le \frac{x^4}{24}.$$

(ii) Now suppose that

$$y^2 = 8 + 2(\cos x - 1).$$

Show that if y > 0,

$$\left| y - \frac{\sqrt{2}}{4} (7 + \cos x) \right| \le \frac{1}{4}.$$

Hint: $y = \sqrt{2}f(z)$ for f as in (i)a) and $z = \cos x - 1$. Use the Taylor polynomial $P_1(z,0)$ and the estimate on the remainder term $R_1(z,0)$

[15 marks]

12. Solve the following differential equations with the given boundary conditions:

(i)
$$y'' + 4y' + 3y = 3x + 1$$

with
$$y(0) = 1$$
, $y'(0) = 2$. (ii)

$$y'' + 4y' + 3y = 5\sin x$$

with
$$y(0) = 1$$
, $y'(0) = 2$.

[15 marks]

13. Consider the function

$$g(x,y) = 3x^2 + 5y^2$$
.

- a) Find the maximum area of a rectangle with horizontal and vertical sides and inscribed in the ellipse g(x, y) = 10.
- b) Find the maximum of the function $f(x,y) = x^2 y^2$ in the region R bounded by the ellipse g(x,y) = 10, including the interior. Any valid methods may be used.

[15 marks]

14.

- a) Find the area of the region R bounded by the parabola $x=y^2$ and the line y=x-2.
 - b) Find the centre of mass $(\overline{x}, \overline{y})$ of R, assuming uniform density.

[15 marks]