

MATH102 Solutions May 2005
Section A

1. The Taylor series of $f(x) = x^{-1} = (1 + (x - 1))^{1/2}$ is

$$1 + \frac{1}{2}(x - 1) + \frac{\frac{1}{2} \cdot \frac{-1}{2}}{2!}(x - 1)^2 \dots = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (x - 1)^n,$$

where

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2} \cdot \frac{-1}{2} \cdot \dots \cdot (\frac{1}{2} - n + 1)}{n!}.$$

This can also be worked out by computing all derivatives of f at $x = 1$.

[3 marks]

a) When $x = 0.5$ the series is convergent and equal to $f(0.5)$.

[1 mark]

b) When $x = 3$ the series is not convergent and it does not even make sense to say that it is equal to $f(3)$.

[1 mark]

No explanation is required in a) or b).

5 = 3 + 1 + 1 marks

2(i) Separating the variables, we have

$$\int \frac{dy}{y^2} = \int \frac{dx}{x},$$

$$-\frac{1}{y} = \ln|x| + C,$$

or

$$y = -\frac{1}{C + \ln|x|}.$$

2(ii) Using the integrating factor method, we have

$$\frac{dy}{dx} - \frac{y}{x} = 1,$$

and the integrating factor is

$$\exp\left(-\int \frac{dx}{x}\right) = \exp \ln x^{-1} = x^{-1}.$$

So the equation becomes

$$\frac{d}{dx} \frac{y}{x} = \frac{1}{x}.$$

Integrating gives

$$\frac{y}{x} = \ln|x| + C.$$

So the general solution is

$$y = x \ln |x| + Cx.$$

3 marks for (i) 4 marks for (ii).

[7 marks]

3. Try $y = e^r x$. Then

$$r^2 + 4r - 5 = 0 \Rightarrow (r - 1)(r + 5) = 0 \Rightarrow r = -5 \text{ or } r = 1.$$

So the general solution is

$$y = Ae^x + Be^{-5x}.$$

[2 marks]

So $y' = Ae^x - 5Be^{-5x}$ and the initial conditions $y(0) = 1$, $y'(0) = -1$ give

$$A + B = 1, \quad A - 5B = -1 \rightarrow 6B = 2, \quad A = 1 - B \Rightarrow B = \frac{1}{3}, A = \frac{2}{3}.$$

So

$$y = \frac{2}{3}e^x + \frac{1}{3}e^{-5x}.$$

[3 marks]

2 + 3 = 5 marks

4. We have

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2 + x^3}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{x^2 + y^2} + \lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} = 1 + 0,$$

because $|x^3| \leq |x| \cdot x^2 \leq |x|(x^2 + y^2)$.

[3 marks] For the second one, $x^4 + y^4 \leq (x^2 + y^2)^2$ and so

$$\frac{x^4 + y^4}{x^2 + y^2} \geq \frac{1}{x^2 + y^2}$$

and $x^2 + y^2 \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$. So the limit does not exist.

[2 marks]

[3 + 2 = 5 marks]

5.

$$\frac{\partial f}{\partial x} = 3x^2y - y^3, \quad \frac{\partial f}{\partial y} = x^3 - 3xy^2,$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy,$$

$$\frac{\partial^2 f}{\partial y \partial x} = 3x^2 - 3y^2,$$

$$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 - 3y^2,$$

so that these last two are equal, and

$$\frac{\partial^2 f}{\partial y^2} = -6xy,$$

[4 marks] which gives

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

[1 mark]

[5 marks]

6. By the Chain Rule,

$$\frac{\partial f}{\partial x} = F'(x + ct) + G'(x - ct), \quad \frac{\partial f}{\partial t} = cF'(x + ct) - cG'(x - ct).$$

So

$$\frac{\partial^2 f}{\partial x^2} = F''(x + ct) + G''(x - ct), \quad \frac{\partial^2 f}{\partial t^2} = c^2 F''(x + ct) + c^2 G''(x - ct),$$

and so

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2},$$

as required.

[5 marks]

7. For

$$f(x, y, z) = x^3 y - y^2 z + 2xyz,$$

we have

$$\nabla f(x, y, z) = (3x^2 y + 2yz)\mathbf{i} + (x^3 - 2yz + 2xz)\mathbf{j} + (2xy - y^2)\mathbf{k}$$

So

$$\nabla f(x, y, z)(1, 1, 1) = 5\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

2 marks

The vector $(-1, 1, 2) = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ has length $\sqrt{6}$ So the directional derivative in the direction $(-1, 1, 2)$ is

$$\frac{1}{\sqrt{6}}(5\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \frac{-2}{\sqrt{6}}.$$

[3 marks]

[2 + 3 = 5 marks.]

8. For

$$f(x, y) = x^2 + x^2y + 2y^2,$$

we have

$$\frac{\partial f}{\partial x} = 2x + 2xy, \quad \frac{\partial f}{\partial y} = x^2 + 4y.$$

[2 marks]

So at a stationary point,

$$x(1+y) = 0 = x^2 + 4y \Rightarrow x = y = 0 \text{ or } (y = -1 \text{ and } x^2 = 4) \Rightarrow (x, y) = (0, 0) \text{ or } (2, -1) \text{ or } (-2, -1).$$

[2 marks]

$$A = \frac{\partial^2 f}{\partial x^2} = 2 + 2y, \quad B = \frac{\partial^2 f}{\partial y \partial x} = 2x, \quad C = \frac{\partial^2 f}{\partial y^2} = 4.$$

For $(x, y) = (0, 0)$ we have $A = 2 > 0$, $B = 0$, $C = 4$ and $AC - B^2 = 8 > 0$. So $(0, 0)$ is a minimum.

For $(x, y) = (\pm 2, -1)$ $A = 0$, $B = \pm 4$, $C = 4$. So $AC - B^2 = -16 < 0$, and both these points are saddles.

[4 marks]

[2 + 2 + 4 = 8 marks]

9. For

$$f(x, y) = \frac{1}{x^2 + y},$$

we have

$$\frac{\partial f}{\partial x} = \frac{-2x}{(x^2 + y)^2}, \quad \frac{\partial f}{\partial y} = \frac{-1}{(x^2 + y)^2}.$$

So

$$f(1, 1) = \frac{1}{2}, \quad \frac{\partial f}{\partial x}(1, 1) = -\frac{1}{2}, \quad \frac{\partial f}{\partial y}(1, 1) = -\frac{1}{4}.$$

So the linear approximation is

$$\frac{1}{2} - \frac{1}{2}(x - 1) - \frac{1}{4}(y - 1).$$

[It would be acceptable to realise that

$$\begin{aligned} f(x, y) &= (2 + 2(x - 1) + (x - 1)^2 + (y - 1))^{-1} \\ &= \frac{1}{2}(1 + (x - 1) + \frac{1}{2}(x - 1)^2 + \frac{1}{2}(y - 1))^{-1} \end{aligned}$$

and to expand out.]

[4 marks]

10.

$$\int \int_R x^2 dy dx = \int_{-2}^2 \int_0^{4-x^2} x^2 dy dx$$

$$\begin{aligned}
&= \int_{-2}^2 [x^2 y]_0^{4-x^2} dx = \int_{-2}^2 (4x^2 - x^4) dx \\
&= \left[\frac{4x^3}{3} - \frac{x^5}{5} \right]_{-2}^2 = 2 \left(\frac{32}{3} - \frac{32}{5} \right) = \frac{128}{15}.
\end{aligned}$$

[6 marks]

Section B

11. (i) We have $f^{(2k)}(x) = (-1)^k \cos x$ and $f^{(2k-1)}(x) = (-1)^k \sin x$. So $f^{(2k)}(0) = (-1)^k$ and $f^{(2k-1)}(0) = 0$. So:

a) $P_3(x) = 1 - \frac{1}{2}x^2$ and $R_3(x) = \frac{1}{4!}x^4 \cos c$ for some c between 0 and x .

b) For any $k \geq 1$

$$P_{2k-1}(x) = \sum_{r=0}^{k-1} (-1)^r \frac{x^{2r}}{(2r)!}, \quad R_{2k-1}(x) = (-1)^k \frac{x^{2k}}{(2k)!} \cos c$$

for some c between 0 and x .

[6 marks]

Since $|\cos c| \leq 1$ for all c , and $\cos c \geq 0$ for $c \in [-\frac{\pi}{2}, 1]$ we have

$$0 \leq R_3(x) \leq \frac{x^4}{(4)!} \cos c \leq \frac{x^4}{(4)!}.$$

So for $x \in [-1, 1]$

$$0 \leq R_3(x) \leq \frac{x^4}{24}.$$

So, since $\cos x = P_3(x) + R_3(x)$,

$$1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24},$$

and

$$1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{11x^2}{24}.$$

[3 marks]

(ii) Using the Taylor series of $\cos x$ at 0:

a)

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{1 - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} \dots\right)}{x^2} = \lim_{x \rightarrow 0} \frac{\frac{x^2}{2} - \frac{x^4}{24} \dots}{x^2} \\
&= \lim_{x \rightarrow 0} \frac{\frac{1}{2} - \frac{x^2}{24} \dots}{1} = \frac{1}{2};
\end{aligned}$$

[3 marks]

b)

$$\lim_{x \rightarrow 0} \left(\frac{1}{x^2} - \frac{1}{2(1 - \cos x)} \right) = \lim_{x \rightarrow 0} \frac{2(1 - \cos x) - x^2}{2x^2(1 - \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{-\frac{x^4}{12} + \dots}{x^4 - \frac{x^6}{12} \dots} = \frac{-1}{12}.$$

[3 marks]

[6 + 3 + 3 + 3 = 15 marks.]

12. For the complementary solution in both cases, if we try $y = e^{rx}$ we need

$$r^2 - 2r + 1 = (r - 1)^2 = 0,$$

that is, $r = 1$. So the complementary solution is $Axe^x + Be^x$.

[3 marks]

(i) We try $y_p = Ce^{-x}$. Then $y'_p = -Ce^{-x}$ and $y''_p = Ce^{-x}$. So $y''_p - 2y'_p + y_p = 4Ce^{-x}$. So $C = 1$. So the general solution is

$$y = Axe^x + Be^x + e^{-x}$$

[3 marks]

This gives

$$y' = A(x+1)e^x + Be^x - e^{-x}.$$

So putting $x = 0$ the boundary conditions give

$$B = 0, \quad A + B = 0 \quad \Rightarrow \quad B = 0, \quad A = 0.$$

So the solution is

$$e^{-x}$$

[3 marks]

(ii) We try $y_p = Cx + D$. Then $y'_p(x) = C$ and $y''_p = 0$. So $y''_p - 2y'_p + y_p = -2C + Cx + D = x$. So $C = 1$ and $D = 2$. So the general solution is

$$y = Axe^x + Be^x + x + 2$$

[3 marks] This gives

$$y'(x) = A(1+x)e^x + Be^x + 1$$

So $y(0) = 1$, $y'(0) = -1$ give

$$B + 2 = 1, \quad A + B = -2,$$

so $B = -1$ and $A = -1$ and

$$y = -xe^x - e^x + x + 2.$$

[3 marks]

[5 × 3 = 15 marks]

13. For $f(x, y) = 3x^2 + x^2y + y^2$, $g(x, y) = 2x^2 + y^2$, we have

$$\nabla f = (6x + 2xy)\mathbf{i} + (x^2 + 2y)\mathbf{j}$$

$$\nabla g = 4x\mathbf{i} + 2y\mathbf{j}.$$

[2 marks]

At a stationary point of f , we have

$$2x(3 + y) = x^2 + 2y = 0.$$

So $x = 0$ or $y = -3$. If $x = 0$ then $y = 0$. If $y = -3$ then $x^2 = 6$, so $x = \pm\sqrt{6}$. So the stationary points are $(0, 0)$ and $(\pm\sqrt{6}, -3)$. Only the first of these satisfies $g(x, y) \leq 1$. So $(0, 0)$ is the only stationary point which is a candidate for a maximum or minimum of f on the set where $g \leq 1$.

[4 marks]

At a maximum or minimum on the set where $g = 1$, we must have $\nabla f = \lambda \nabla g$, that is

$$6x + 2xy = 4\lambda x, \quad x^2 + 2y = 2\lambda y.$$

[1 mark]

The first equation gives $x = 0$ or $3 + y = 2\lambda$. If $x = 0$ then the equation $g = 1$ gives $y^2 = 1$ and $y = \pm 1$.

If $2\lambda = 3 + y$, then plugging into the second equation gives

$$x^2 + 2y = y(3 + y)$$

So multiplying by 2 and replacing $2x^2$ by $1 - y^2$ gives

$$1 + 4y - y^2 = 6y + 2y^2.$$

So

$$3y^2 + 2y - 1 = (3y - 1)(y + 1) = 0.$$

So $y = -1$ or $y = \frac{1}{3}$ and using $g = 1$ gives

$$(x, y) = (0, -1) \text{ or } \left(\pm\frac{2}{3}, \frac{1}{3}\right).$$

So altogether the points on $g = 1$ which can be maxima or minima of f on $g \leq 1$ are

$$(0, 0) \text{ or } (0, \pm 1) \text{ or } \left(\pm\frac{2}{3}, \frac{1}{3}\right).$$

[6 marks]

We have

$$f(0, 0) = 0, \quad f(0, \pm 1) = 1, \quad f\left(\pm\frac{2}{3}, \frac{1}{3}\right) = \frac{43}{27}.$$

So the minimum value of f on $g \leq 1$ is 0, realised at $(0, 0)$, and the maximum is $\frac{43}{27}$, realised at $(\pm\frac{2}{3}, \frac{1}{3})$.

[2 marks.]

[2 + 4 + 1 + 6 + 2 = 15 marks.]

14. The line $x + 2y = 1$ meets the y -axis $x = 0$ at $y = \frac{1}{2}$ and the x -axis $y = 0$ at $x = 1$. So the mass is given by

$$\begin{aligned} M &= \int_0^{1/2} \int_0^{1-2y} xy dx dy = \int_0^{1/2} \frac{y(1-2y)^2}{2} dy \\ &= \left[\frac{y^2}{4} - 2\frac{y^3}{3} + \frac{y^4}{2} \right]_0^{1/2} = \frac{1}{16} - \frac{1}{12} + \frac{1}{32} = \frac{1}{96}. \end{aligned}$$

[5 marks]

Then the centre of mass is $(\bar{x}, \bar{y})$ where

$$\begin{aligned} \bar{x} &= \frac{1}{M} \int_0^{1/2} \int_0^{1-2y} x^2 y dx dy = 96 \int_0^{1/2} \frac{y}{3} (1-2y)^3 dy \\ &= 96 \int_0^{1/2} \left(\frac{y}{3} - 2y^2 + 4y^3 - 8\frac{y^4}{3} \right) dy = 96 \left[\frac{y^2}{6} - \frac{2y^3}{3} + y^4 - 8\frac{y^5}{15} \right]_0^{1/2} \\ &= 96 \left(\frac{1}{24} - \frac{1}{12} + \frac{1}{16} - \frac{1}{60} \right) = -4 + 6 - \frac{8}{5} = \frac{2}{5}, \end{aligned}$$

[5 marks]

Changing the order of integration,

$$\begin{aligned} \bar{y} &= 96 \int_0^{1/2} \frac{y^2}{2} (1-4y+4y^2) dx dy \\ &= 96 \left[\frac{y^3}{6} - \frac{y^4}{2} + 2\frac{y^5}{5} \right]_0^{1/2} = 2 - 3 + \frac{6}{5} = \frac{1}{5}. \end{aligned}$$

So the centre of mass is

$$\left(\frac{2}{5}, \frac{1}{5} \right).$$

[5 marks]

[5 + 5 + 5 = 15 marks]