SECTION A

1. Write down the Taylor series about $x = 1$ for the function

$$f(x) = \frac{1}{x}.$$

State whether this Taylor series is equal to $f(x)$ for:
a) $x = 2$, b) $x = 1.5$. [5 marks]

2. Find the general solutions of the following differential equations:
(i) $x\frac{dy}{dx} - y = 0$,
(ii0 $x\frac{dy}{dx} - y = 1$.

[7 marks]

3. Solve the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 0$$

with the initial conditions $y(0) = 1$, $y'(0) = 2$.

[5 marks]

4. Show, by taking limits along two different paths to the origin $(0, 0)$, that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

does not exist.

[4 marks]

5. Work out all first and second partial derivatives of

$$f(x,y) = \ln(x^2 + y^2),$$

and verify that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

and

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

[6 marks]

6. Use the Chain Rule to work out $\dfrac{\partial F}{\partial u}$ and $\dfrac{\partial F}{\partial v}$, where

$$F(u,v) = f(x(u,v), y(u,v)), \quad f(x,y) = x^2 + y^3, \quad x(u,v) = u + v, \quad y = u - v.$$

[5 marks]

7. Find the tangent plane at the point $(1,1,1)$ to the surface

$$3x^2 - 2xyz + z^2 y = 2.$$

[5 marks]

8. Locate and classify all stationary points of the function

$$f(x,y) = 2y^3 + 6xy + x^2.$$

[8 marks]

9. Find the linear approximation near $(x,y) = (1,1)$ to the function

$$f(x,y) = \frac{1}{x^2 + y^2}.$$

[4 marks]

10. Let T be the triangle in the plane bounded by the lines

$$y = 0, \quad x = 1, \quad x = y.$$

Work out the double integral

$$\int\int_T (x-y)dxdy.$$

You may change the order of integration if you prefer.

[6 marks]

SECTION B

11. Let $f(y) = (1+y)^{-1/2}$. Find the Taylor polynomial $P_2(y)$ near $y = 0$ and the associated error term $R_2(y)$. Hence, or otherwise, show that, for any x,

$$\left|(1+x^2)^{-1/2} - P_2(x^2)\right| \leq \frac{5x^6}{16}.$$

Now work out

$$\int_0^{1/2} P_2(x^2)dx$$

and compare the answer with $\ln((1+\sqrt{5})/2)$, worked out on your calculator. Explain why you expect the two answers to be close, by working out

$$\frac{d}{dx}(\ln(x+\sqrt{1+x^2}))$$

or otherwise.

[15 marks]

12. Solve the following differential equations with the given boundary conditions:

(i)

$$y'' - 4y = x$$

with $y(0) = 1$, $y'(0) = -1$,

(ii)

$$y'' - 4y = \sin x$$

with $y(0) = 1$, $y'(0) = -1$.

[15 marks]

13.

a) Find the minimum distance from the origin $(0,0)$ to the surface

$$g(x,y) = x^2 - 2y^2 = 1$$

b) Find the minimum distance from the origin $(0,0)$ to the surface

$$h(x,y) = 4xy - 3y^2 = 1$$

[*Hint*: consider the function $f(x,y) = x^2 + y^2$ in both cases.]

[15 marks]

14. Find the centre of mass of the plane triangle bounded by the lines

$$x = 0, \quad y = 0, \quad x + 2y = 1,$$

where mass is uniformly distributed.

[15 marks]