PAPER CODE NO.
MATH014

EXAMINER:
DEPARTMENT: TEL. NO

THE UNIVERSITY *of* LIVERPOOL

AUGUST/SEPTEMBER 2005 EXAMINATIONS

Bachelor of Engineering: Foundation Year
Bachelor of Science: Foundation Year
Bachelor of Science: Year 1
No qualification aimed for: Year 1

CALCULUS II and APPLICATIONS TO MECHANICS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

You may attempt all questions. All answers to Section A and the best THREE questions from Section B will be taken into account. Section A carries 55% of the available marks. The marks shown against the sections indicate their relative weights.

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SECTION A

1. Evaluate the following indefinite integrals

(i) $\int (2x^2/3)\,dx$ [2 marks] (ii) $\int \frac{dx}{\sqrt{x}}$ [2 marks]

(iii) $\int \sin(2x-3)\,dx$ [3 marks] (iv) $\int e^{x/3}\,dx$ [3 marks]

2. Evaluate the following definite integrals

(i) $\int_0^2 \frac{dx}{x+1}$ [3 marks] (ii) $\int_{\pi/3}^{\pi/2} \cos(3x)\,dx$ [3 marks]

(iii) $\int_1^2 x^2(1-x)\,dx$ [2 marks] (iv) $\int_0^1 (e^x - e^{-x})\,dx$ [2 marks]

3. Using partial fractions, the following rational functions can be written as

(a) $\frac{x}{(2x-1)(x+2)} = \frac{A}{2x-1} + \frac{B}{x+2}$

(b) $\frac{2}{x(x^2+1)} = \frac{C}{x} + \frac{Dx+E}{x^2+1}$

Compute the constants A, B, C, D, E. [3 marks]

Hence evaluate the following integrals

(i) $\int \frac{x\,dx}{(2x-1)(x+2)}$ [3 marks]

(ii) $\int_1^2 \frac{2\,dx}{x(x^2+1)}$ [4 marks]

4. Use integration by parts to show that

$$\int_0^1 x\,\mathrm{e}^{-2x}\,dx = \frac{\mathrm{e}^2 - 3}{4\mathrm{e}^2}. \quad [7 \text{ marks}]$$

5. Solve the following first order differential equations

(i) $\dfrac{dy}{dx} = 2x,$ [3 marks]

(ii) $\dfrac{dy}{dx} = 3y,$ given that $y = 1$ when $x = -1/3$. [5 marks]

6. Solve the following second order differential equations

(i) $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0,$ [4 marks]

(ii) $\dfrac{d^2y}{dx^2} = -4y,$ given that $y = 1$ when $x = 0$ and $\dfrac{dy}{dx} = 1$ when $x = 0$. [6 marks]

SECTION B

7. A particle of mass m is thrown vertically upwards from the ground by a machine with an initial speed of $30\,\mathrm{m\,s^{-1}}$.

Using the notation that the vertically upwards direction is denoted by y, show that the differential equation governing the motion of the particle is

$$\frac{d^2y}{dt^2} = -g,$$

where g is the gravitational acceleration and t is time.

[4 marks]

Now, obtain the solution $y(t)$ by solving the differential equation above.

[2 marks]

Hence find the maximum height reached by the particle, assuming that the gravitational acceleration g is approximately $10\,\mathrm{m\,s^{-2}}$.

[5 marks]

Also find the height reached by the particle (on its way up from the ground) when its speed is $20\,\mathrm{m\,s^{-1}}$.

[4 marks]

8.

(i) Solve the following second order differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{1}{4}y = 0,$$

given that $y = 1/\mathrm{e}$ when $x = 2$ and $y = 2/\mathrm{e}^2$ when $x = 4$.

[8 marks]

(ii) Evaluate the following definite integral

$$\int_0^{\pi/4} \tan^2(x)\,dx$$

Hint: You may use the substitution $u = \tan(x)$.

[7 marks]

9. A 66 kg cyclist is on level ground on a 6 kg bicycle. It is a fairly windy day and the resistance felt by the cyclist is proportional to his speed. Show that the differential equation governing the motion of the cyclist is

$$m\frac{dv}{dt} = -kv,$$

where v is the cyclist's speed, t is time and k is a positive constant.

[2 marks]

Solve the differential equation above, given that the initial speed of the cyclist is $7\,\mathrm{m\,s^{-1}}$ and the constant $k = 3\,\mathrm{kg\,s^{-1}}$.

Hint: You should obtain a formula in the form $v(t) = A\,\mathrm{e}^{b\,t}$, where A and b are constants to be found.

[5 marks]

Using the equation obtained, how far would the cyclist travel before reaching a complete stop, given that at time $t = 0$ the distance travelled is zero?

Hint: Consider what happens to the exponential function as t gets larger?

[4 marks]

Approximately how long would it take for the cyclist's speed to drop to $3\,\mathrm{m\,s^{-1}}$?

[4 marks]

10. A simple pendulum of length $L = 0.9\,\mathrm{m}$ makes an angle θ with the vertical. As it swings back and forth, its motion is approximately described by the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\,\theta = 0,$$

where t is time and $g = 10\,\mathrm{m\,s^{-2}}$ is the gravitational acceleration.

Given that, at $t = 0$, $\theta = 1/2$ and $\dfrac{d\theta}{dt} = \dfrac{5\sqrt{3}}{3}\,\mathrm{s^{-1}}$, solve the differential equation above.

[5 marks]

Also show by substitution that

$$\theta(t) = \cos(10t/3 - \pi/3)$$

satisfies the differential equation, and the initial conditions above.

[5 marks]

Plot this function clearly on a graph, where the horizontal axis is t and the vertical axis is θ, for $0 \le t \le \pi$. Hence, find the period of the solution.

[5 marks]