PAPER CODE NO.
MATH 013

THE UNIVERSITY *of* LIVERPOOL

JANUARY 2004 EXAMINATIONS

Bachelor of Engineering : Foundation Year
Bachelor of Science : Foundation Year

MATHEMATICAL METHODS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

You may attempt all questions. All answers to Section A and the best THREE answers to Section B will be taken into account.
Numerical answers should be given correct to four places of decimals.

SECTION A

1. Determine the radian measure of the angle α of 420^0, expressed as a rational multiple of π. The formula for $\sin(A+B)$ states that

$$\sin(A+B)=\sin(A)\cos(B)+\cos(A)\sin(B).$$

Using this formula or otherwise find the exact value for $\sin(\alpha)$, *without using tables or a calculator*. (Show all your working.)
Hence determine all the angles θ, in the range $[-2\pi, 2\pi]$ satisfying $\sin(\theta)=\sin(\alpha)$. Your answers should be expressed in radian measure.

[6 marks]

2. Sketch the graph of $y=\tan(x)$ in the range $-2\pi \le x \le 2\pi$. Determine numerically the solutions of $\tan(x)=2$ in the same range. Express, if possible, your results in radian measure.

[9 marks]

3. Find the domain of x for which *both* the functions $\log_3(x)$ and $\log_3(2x+5)$ are defined. Solve the equation

$$\log_3(2x+5)-\log_3(x)=1.$$

[7 marks]

4. You are given the values of $\log_e(32)=3.465736$ and $\log_e(16)=2.772589$, correct to six decimal places. Obtain the values of the following

$$\log_e(512), \qquad \log_e(2), \qquad \log_e(64),$$

without using tables or a calculator, correct to four decimal places. (Show all your working. HINT: for second part use $8^3=512$.)

[6 marks]

5. Write down the first five rows of Pascal's triangle. Hence or otherwise find the coefficient of x^6 in the expansion of

$$(2x^2-1)^4.$$

[6 marks]

6. Let $q(x)$ be the quadratic function $q(x)=4-3x-x^2$. Determine the zeros of $q(x)$ and the position and nature of its turning point. Hence sketch the graph of $q(x)$.

[7 marks]

7. Express the rational function $f(x)$ in partial fractions, where

$$f(x)=\frac{4x-7}{(x-3)(x-4)}.$$

[5 marks]

8. Express the complex number

$$z=\frac{8-\mathrm{i}}{4-3\mathrm{i}}$$

in the form $z=a+b\mathrm{i}$.
Determine numerically the modulus and argument of z. The argument should, preferably, be expressed in radian measure. Hence, or otherwise, find the modulus and argument of z^2.

[9 marks]

SECTION B

9. Find two values of θ between 0 and π radians satisfying the equation

$$6\sin^2(\theta) = 5 + \cos(\theta).$$

[7 marks]

Using the identity $\cos(A) - \cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$ or otherwise, find the range of values of a for which the equation

$$\cos(x + 90^0) - \cos(x) = a\,,$$

has real solutions. For the case $a = 1/2$, find all the solutions in the interval $0^0 \le x \le 360^0$.

[8 marks]

10. (i) On separate diagrams sketch the curves $y = 2e^x$ for real x, and $y = \log_e(x^2)$ for $x > 0$ and $x < 0$.

[4 marks]

(ii) Solve the following equations:

$$\log_4(x) = 3, \qquad \log_y(125) = 3\,.$$

[4 marks]

(iii) A marathon runner sets out to cover a 26.2 mile course at an initial speed of 12 miles per hour. As he tires his running speed S declines according to the formula

$$S = \alpha - e^{kt}\,,$$

where t is the time in hours, and α and k are constants. Show that $\alpha = 13\text{mph}$. After 1hour his speed has declined by 5%. Determine the value of k. If his overall average speed for the whole course is 10.48mph, calculate both the time he takes to complete the course and his running speed at the finish.

[7 marks]

11. (i) If α and β are the roots of the equation $-2x^2-6x+3=0$, write down the values of a) $\alpha\beta$, b) $\alpha+\beta$, c) $\alpha^2+\beta^2$ and d) $(\alpha-\beta)^2$, *without determining the values of α and β* individually.

[8 marks]

(ii) Plot a table of the values of the following cubic polynomial

$$p(x)=-x^3+3x^2+2x-2,$$

for $x=-2,-1,0,1,2,3$ and 4. Sketch the curve of the polynomial, and find all the roots of $p(x)=0$.

[7 marks]

12. (i) A complex number z has modulus one and argument $\pi/6$. Express each of the following complex numbers in the form $a+b\mathrm{i}$:

$$z,\ z^2,\ z^3,\ \frac{1}{z},$$

and plot them on the Argand diagram.

[10 marks]

(ii) Find the real values of a and b such that

$$\frac{a+b\mathrm{i}}{2+2\mathrm{i}}=-\mathrm{i}\,.$$

[5 marks]