PAPER CODE NO.
MATH012

THE UNIVERSITY *of* LIVERPOOL

SUMMER 2006 EXAMINATIONS

Bachelor of Engineering : Foundation Year
Bachelor of Science : Foundation Year
Bachelor of Science : Year 1
Bachelor of Science : Year 2

VECTORS AND INTRODUCTION TO STATISTICS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B.
The total of the marks available on Section A is 55.
i, **j** and **k** are unit vectors along the x, y and z axes respectively.

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SECTION A

1. Let $ABCD$ be a parallelogram. Given that $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{BC} = \mathbf{v}$, express each of the following in terms of $\mathbf{u}$ and $\mathbf{v}$:

   (a) $\overrightarrow{CD}$

   (b) $\overrightarrow{CA}$

   (c) $\overrightarrow{BP}$, where P is the mid-point of $\overrightarrow{CD}$.

   [5 marks]

2. The points P, Q and R have Cartesian coordinates $(2, 4, 3)$, $(-2, 2, 1)$ and $(2, 1, -1)$ respectively, where lengths are measured in metres.

   Find:

   (a) the lengths of the sides of triangle PQR, correct to the nearest centimetre

   (b) $\overrightarrow{QP} \cdot \overrightarrow{RP}$

   (c) the angle $\angle QPR$ in degrees (to the nearest 0.1degree).

   (d) the coordinates of the point S such that $PQRS$ is a parallelogram.

   [13 marks]

3. Let $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

   (a) Compute $3\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$.

   (b) Compute $(3\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}$ and $(3\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - 2\mathbf{v})$.

   (c) Find a unit vector parallel to $\mathbf{u} \times \mathbf{v}$.

   [8 marks]

4. The points A and B have Cartesian coordinates $(1, 2, 3)$ and $(3, 3, 1)$ respectively.

   (a) Compute $\overrightarrow{AB}$

   (b) Find the vector equation of the line $\mathcal{L}$ through A and B

   (c) Suppose point P has position vector $\mathbf{p} = -\mathbf{i} + 6\mathbf{k}$. What is the vector from the point P to a point R on the line $\mathcal{L}$?

   (d) Compute the shortest distance from P to the line $\mathcal{L}$.

[**8 marks**]

5. Let O be the origin of co-ordinates. A particle P moves so that its position vector $\mathbf{r}$ with respect to O at time t is given by

$$\mathbf{r} = (2t - 1)\mathbf{i} + t\sin(2t)\mathbf{j} + 2t^2\mathbf{k}$$

   where t is measured in seconds, distances are measured in metres and $\sin(2t)$ is evaluated by treating $2t$ seconds as $2t$ radians. Find:

   (a) the position of P at time $t = 0$ seconds

   (b) the velocity of P at time t seconds

   (c) the speed of P at $t = \frac{\pi}{4}$ seconds, to the nearest cm/sec

   (d) the acceleration of P at $t = 0$ seconds.

[**7 marks**]

6. An aircraft sets out from the origin O. The wind velocity relative to the ground is $\mathbf{w} = 50\mathbf{i}$ km/hr where $\mathbf{i}$ is a unit vector pointing East.

   The aircraft travels at a constant velocity $\mathbf{u} = (-150\mathbf{i} + 400\mathbf{j})$ km/hr relative to the air. Here $\mathbf{j}$ is a unit vector pointing North.

   (a) Give an expression for the velocity $\mathbf{v}$ of the aircraft relative to the ground.

   (b) Hence write down an expression for the position vector of the aircraft at time t hours.

   (c) Find the time in minutes at which the aircraft has flown 160 km North (relative to the ground).

   (d) Find the position vector of the point P the aircraft reaches after it has flown 160 km North (relative to the ground).

   [**6 marks**]

7. Find the volume of the parallelepiped with edges formed by the vectors $3\mathbf{i}+\mathbf{j}-2\mathbf{k}$, $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$, and $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$.

   [**4 marks**]

8. What does the conditional probability $P(X|Y)$ of events X and Y mean?

   Two machines A and B make compact discs (CDs). In a given batch at the factory, 10% of the CDs are made by A and 90% by B. Also, 80% of the CDs made by A are acceptable, and 60% of the CDs made by B are acceptable. What is the probability that a given CD chosen at random from the whole batch is acceptable?

   [**4 marks**]

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SECTION B

9. The four distinct points A, B, C and D are non-collinear and such that $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{BC} = \mathbf{v}$ and $\overrightarrow{DA} = \mathbf{w}$. Suppose that

$$\mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k}, \quad , \quad \mathbf{v} = \mathbf{i} - 3\mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{w} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}.$$

(a) Using the vectors $\mathbf{u}$ and $\mathbf{v}$, show that a unit vector normal to the plane containing the points A, B and C is given by

$$\hat{\mathbf{n}} = \frac{2\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{6}}.$$

(b) Show by explicitly calculating the scalar products that

$$\hat{\mathbf{n}} \cdot \mathbf{u} = 0 \quad \text{and} \quad \hat{\mathbf{n}} \cdot \mathbf{v} = 0.$$

(c) Show that A, B, C and D lie in the same plane.

(d) Suppose that A is the point $(2, 3, 1)$. What is the Cartesian equation of the plane through A, B, C and D?

(e) Find where the straight line

$$\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$$

intersects the plane through A, B, C and D.

[15 marks]

10. Suppose that the line $\mathcal{L}_1$ has vector equation

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

and that the line $\mathcal{L}_2$ has vector equation

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} - \mathbf{k} + \mu(\mathbf{i} - \mathbf{k}).$$

(a) Write down the coordinates of *any* two points on the line $\mathcal{L}_1$.

(b) Determine two unit vectors $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ which are respectively parallel to the lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

(c) Compute the angle between the lines.

(d) Show that the lines intersect and find the coordinates of the point of intersection.

[**15 marks**]

11. The planes Π_1, Π_2 and Π_3 have equations

$$x - y - 4z = 13, \qquad 4x - y - z = 10, \qquad \text{and} \qquad 3x + 2y + z = 5,$$

respectively.

(a) Find a normal to each plane.

(b) Find the angle in degrees between the normals to the planes Π_1 and Π_2.

(c) Find the point of intersection of Π_1, Π_2 and Π_3.

[**15 marks**]

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12. Define the mean, mode and median of a set of values.

The rainfall on 15 consecutive days is measured (in millimetres, mm) as

$$5, 3, 2, 0, 5, 7, 10, 11, 1, 7, 3, 1, 2, 13, 5.$$

(a) Draw a bar chart to show the number of days with rainfalls in the ranges 0-5mm, 6-10mm and 11-15mm.

(b) What is the frequency and relative frequency of a result of 7mm?

(c) What is the mean rainfall in millimetres?

(d) What are the mode and median?

(e) What is the standard deviation of the daily rainfall (to the nearest 0.01 millimetre)?

[**15 marks**]