PAPER CODE NO.
MATH012

THE UNIVERSITY
of LIVERPOOL

SUMMER 2004 EXAMINATIONS

Bachelor of Engineering : Foundation Year
Bachelor of Science : Foundation Year
Bachelor of Science : Year 1
Bachelor of Science : Year 2

VECTORS AND INTRODUCTION TO STATISTICS

TIME ALLOWED : Three Hours

INSTRUCTIONS TO CANDIDATES

Answer ALL questions in Section A and THREE questions from Section B.
The total of the marks available on Section A is 55.

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SECTION A

1. In parallelogram $ABCD$, the sides AB and AD are given by the vectors $\mathbf{u}$ and $\mathbf{v}$ respectively. The points L and M are the midpoints of the sides AB and BC respectively. Find expressions for the following in terms of $\mathbf{u}$ and $\mathbf{v}$:

   (a) $\overrightarrow{AL}$

   (b) $\overrightarrow{LM}$

   (c) $\overrightarrow{DL}$.

[5 marks]

2. The points P, Q and R have Cartesian coordinates $(4, 1, 1)$, $(3, 1, 2)$ and $(5, 2, 1)$ respectively where lengths are measured in centimetres.

   Find:

   (a) $\overrightarrow{PQ}$

   (b) $\overrightarrow{QR}$

   (c) the coordinates of the point S such that $PQRS$ is a parallelogram

   (d) the total length of the sides of parallelogram $PQRS$ in centimetres to the nearest millimetre

   (e) $\overrightarrow{QP} \cdot \overrightarrow{QR}$

   (f) all angles of the parallelogram $PQRS$.

[13 marks]

3. Let $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually orthogonal unit vectors.

   (a) Find $\mathbf{u} + 2\mathbf{v}$ and $2\mathbf{u} - \mathbf{v}$.

   (b) Find $(\mathbf{u} + 2\mathbf{v}) \cdot \mathbf{u}$ and $(\mathbf{u} + 2\mathbf{v}) \cdot (2\mathbf{u} - \mathbf{v})$.

   (c) A unit vector parallel to $\mathbf{u} - \mathbf{v}$.

   (d) $\mathbf{u} \times \mathbf{v}$.

[8 marks]

4. The points A and B have Cartesian coordinates $(1, -1, -1)$ and $(-1, -2, 0)$ respectively. Find:

   (a) $\overrightarrow{AB}$

   (b) the vector equation of the line $\mathcal{L}$ through A and B

   (c) Suppose point P has position vector $\mathbf{p} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. What is the vector from a point R on the line $\mathcal{L}$ to the point P ?

   (d) the point on $\mathcal{L}$ closest to the point P.

[**8 marks**]

5. Let O be a fixed origin and let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be constant, mutually orthogonal unit vectors. A particle P moves so that its position vector $\mathbf{r}$ with respect to O at time t is given by

$$\mathbf{r} = t^2\mathbf{i} + te^{-2t}\mathbf{j} + (3 - t)\mathbf{k}$$

   where t is measured in seconds and distances are measured in metres. Find:

   (a) the position of P at time $t = 0$ seconds

   (b) the velocity of P at time t seconds

   (c) the speed of P at $t = 3$ seconds, to the nearest cm/sec

   (d) the acceleration of P at $t = 0$.

[**6 marks**]

6. An aircraft sets out from the origin O. The wind velocity relative to the ground is $\mathbf{w} = 100\mathbf{i}$ km/hr where $\mathbf{i}$ is a unit vector pointing East.

   The aircraft travels at a constant velocity $\mathbf{u} = (-150\mathbf{i} + 80\mathbf{j})$ km/hr relative to the air. Here $\mathbf{j}$ is a unit vector pointing North.

   (a) Give an expression for the velocity $\mathbf{v}$ of the aircraft relative to the ground.

   (b) Hence write down an expression for the position vector of the aircraft at time t hours.

   (c) Find the time in minutes at which the aircraft has flown 120 km North.

   (d) Find the position vector of the point P the aircraft reaches after it has flown 120 km North.

[6 marks]

7. Evaluate the determinant

$$\begin{vmatrix} 2 & 1 & x \\ 1 & 0 & 4 \\ 2 & x & -1 \end{vmatrix}.$$

   Find the values of x for which the determinant is equal to 18.

[4 marks]

8. What does the conditional probability $P(X|Y)$ of events X and Y mean?

   Two machines A and B make components for cars. In a given batch at the car factory, 60% of components are made by A and 40% by B. Also, 90% of components made by A are acceptable, and 80% of components made by B are acceptable. What is the probability that a given component chosen at random from the whole batch is acceptable?

[5 marks]

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SECTION B

9. The four distinct points A, B, C and D are non-collinear and such that $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{BC} = \mathbf{v}$ and $\overrightarrow{DA} = \mathbf{w}$.

   (a) Find an expression for $\overrightarrow{CD}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

   (b) What condition should be satisfied by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in order that $ABCD$ should be a parallelogram with AB and DC as opposite sides?

   (c) Suppose that, in terms of mutually orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$,

$$\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, \quad , \quad \mathbf{v} = \mathbf{i} - \mathbf{j} \quad \text{and} \quad \mathbf{w} = 4\mathbf{i} - 7\mathbf{j} + 6\mathbf{k}.$$

   (i) Show that $ABCD$ is not a parallelogram.
   (ii) Using the vectors $\mathbf{u}$ and $\mathbf{v}$, show that a unit vector normal to the plane containing the points A, B and C is given by

$$\mathbf{n} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{3}.$$

   (iii) Show by explicitly calculating the scalar products that

$$\mathbf{n} \cdot \mathbf{u} = 0 \quad \text{and} \quad \mathbf{n} \cdot \mathbf{v} = 0.$$

   (iv) Show that A, B, C and D lie in the same plane.
   (v) Suppose A is the point $(2, 1, -1)$. A line is drawn through A perpendicular to the plane containing A, B, C and D. What are the co-ordinates of the two points on the line a distance 1 unit away from A?

[15 marks]

10. Suppose that the line $\mathcal{L}_1$ has vector equation

$$\mathbf{r} = 2\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - 2\mathbf{j} - \mathbf{k})$$

and that the line $\mathcal{L}_2$ has vector equation

$$\mathbf{r} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{i} - 4\mathbf{j} + \mathbf{k})$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors parallel to the coordinate axes Ox, Oy and Oz.

(a) Write down the coordinates of *any* two points on the line $\mathcal{L}_1$.

(b) Determine two unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ which are respectively parallel to the lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

(c) Show that the angle between the lines is 45 degrees.

(d) Show that the lines intersect and find the coordinates of the point of intersection.

[15 marks]

11. The planes Π_1 and Π_2 have equations

$$x + y - 2z = 2 \qquad \text{and} \qquad 2x - y - z = 7$$

respectively, with respect to Cartesian axes $Oxyz$.

(a) Find a normal to each plane.

(b) Find the angle in degrees between the normals to the planes Π_1 and Π_2.

(c) Show that the vector equation of the line $\mathcal{L}$ of intersection of the planes Π_1 and Π_2 can be written in the form

$$\mathbf{r} = -4\mathbf{j} - 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) \ .$$

(d) Find the coordinates of the point A which corresponds to $\lambda = 2$.

(e) Find the Cartesian equation of the plane which is perpendicular to $\mathcal{L}$ and passes through the point A.

[15 marks]

12. Define the mean, mode and median of a set of values.

12 students sit an exam with marks given by

$$6, 4, 2, 5, 9, 4, 1, 14, 9, 10, 11, 9.$$

(a) Draw a bar chart to show the number of students with marks in the ranges 0-5, 6-10 and 11-15.

(b) What is the frequency and relative frequency of a result of 4?

(c) What is the mean mark?

(d) What are the mode and median?

(e) What is the standard deviation of the marks?

[**15 marks**]